Necessity of Additional Balance Law and the Importance of GM/WF in Non-Classical Continuum Theories for Solid Continua

By

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Abstract

In non-classical continuum theories for solid continua incorporating the internal rotations due to displacement gradient tensor and conjugate moment tensor in the mathematical description of the deformation physics, the fundamental question "*are the conservation and balance laws derived for classical continuum theories sufficient to ensure equilibrium of the deforming solid matter*" is addressed in the present work.

It is shown that in the non-classical theories considered here for solid continua, an additional balance law, balance of moments of moments is required to ensure equilibrium of the deforming matter due to presence of additional physics of internal rotations and conjugate moment tensor that are absent in the classical continuum theories. Thermoelastic non-classical solids with small deformation and small strain is used as an example to derive theoretical details as well as to present three model problems studies in plane non-classical elasticity.

The second part of the research presented here considers finite element processes based on GM/WF for non-classical solid continua with small strain and small deformation in which the deformation due to mechanical work is reversible. Using plane non-classical elasticity as an example, it is shown that when the conservation and balance laws are cast purely in terms of displacements in Lagrangian description, the balance of linear momenta results in fourth order partial differential equations in displacements in which adjoint $\boldsymbol{A}^*$ of the differential operator $\boldsymbol{A}$ is same as $\boldsymbol{A}$. A finite element formulation of the PDEs in displacements is constructed using GM/WF in which the integral form is variationally consistent and is compared with least squares finite element formulation constructed for a first order system of PDEs in which the integral form is

also variationally consistent. The meritorious features and advantages of GM/WF over least squares process are demonstrated. Model problem studies are also presented to illustrate these features.

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$\boldsymbol{J}$: Jacobian of deformation

${}^{d}\boldsymbol{J}$: Displacement gradient tensor

${}^{d}_{s}\boldsymbol{J}$: Symmetric part of displacement gradient tensor

${}^{d}_{a}\boldsymbol{J}$: Antisymmetric part of displacement gradient tensor

${}^{\Theta}\boldsymbol{J}$: Rotation gradient tensor

${}^{\Theta}_{s}\boldsymbol{J}$: Symmetric part of rotation gradient tensor

${}^{\Theta}_{a}\boldsymbol{J}$: Antisymmetric part of rotation gradient tensor

${}_{i}\boldsymbol{\Theta}$: Rotation tensor

$\boldsymbol{\sigma}$: Cauchy stress tensor

${}_{s}\boldsymbol{\sigma}$: Symmetric part of Cauchy stress tensor

${}_{a}\boldsymbol{\sigma}$: Antisymmetric part of Cauchy stress tensor

$\boldsymbol{m}$: Cauchy moment tensor

${}_{s}\boldsymbol{m}$: Symmetric part of Cauchy moment tensor

${}_{a}\boldsymbol{m}$: Antisymmetric part of Cauchy moment tensor

$\boldsymbol{\varepsilon}$: Strain tensor

θ : Absolute temperature

$\boldsymbol{q}$: Heat vector (Lagrangian description)

Φ : Helmholtz free energy density (Lagrangian description)

Ψ : Gibbs potential (Lagrangian description)

$_s\sigma_{xx}$: Normal stress in x direction

$_s\sigma_{yy}$: Normal stress in y direction

$_s\sigma_{yx}$: Symmetric shear stress

$_a\sigma_{yx}$: Antisymmetric shear stress

E : Modulus of elasticity

ν : Poisson's ratio

u : Displacement in x direction (dimensionless)

v : Displacement in y direction (dimensionless)

ρ_0 : Density in reference configuration

L_0 : Length in reference configuration

h : Characteristic length in space

p : Degree of approximation in space

k : Order of approximation in space

Chapter 1

Introduction and scope of work

In classical continuum theories for solid matter, the symmetric part of the Jacobian of deformation (${}_s\boldsymbol{J}$) or the symmetric part of the symmetric part of the displacement gradient tensor (${}_s^d\boldsymbol{J}$) is considered to form the basis for deriving the conservation and balance laws [1]. Decomposition of these tensors into their symmetric and skew symmetric components or their polar decomposition into pure strech and rotation tensors shows that these tensors consists of strech (strain) as well as pure rotation. Classical constinuum theories only consider strech or strain part of rotation tensors and the rotation part is ignored completely. Thus, classical continuum theories do not incorporate $\boldsymbol{J}$ or ${}^d\boldsymbol{J}$ in their entirety in the conservation and the balance laws. In deforming solid matter $\boldsymbol{J}$ or ${}^d\boldsymbol{J}$ vary between a material point and its neighbors, hence, so does the rotation tensor. For no slip between the neighboring material points these rotations must be resisted by the matter, thus creating conjugate moment tensor. This physics exists in all deforming continua, but is neglected in the classical continuum theories.

Surana et al. [2–5] have presented non-classical theories for solid incorporating $\boldsymbol{J}$ or ${}^d\boldsymbol{J}$ in their entirety in the derivation of conservation and balance laws. The authors have modified the current continuum theories for solid continua that already contains $\boldsymbol{J}$ or ${}^d\boldsymbol{J}$ to additionally incorporate ${}_a\boldsymbol{J}$ or $a^d\boldsymbol{J}$ (antisymmetric part of $\boldsymbol{J}$ or ${}^d\boldsymbol{J}$) and the conjugate moment tensor. These new continuum theories are called non-classical continuum theories incorporating internal rotation, hence incorporating $\boldsymbol{J}$ or ${}^{d}\boldsymbol{J}$ in their entirety. Surana et al. [2–5] have shown that conservation of mass, balance of linear momenta, balance of angualar momenta, first and second law of thermodynamics used in classical continuum theories also remain (with some modifications and/or additions) valid in case of non-classical continuum theories. Yang etc. [6] were the first ones to point out that in non-classical continuum theories in addition to conservation and balance laws used in continuum theories additional balance law, "balance of moments of moments" is needed to ensure equilibrium of the deforming solid. The authors presented reasoning for the need of their balance law based on geometric considerations. This additional balance law has been used by Surana et al. [3–5, 7, 8] in all of their works on non-classical continuum theories. In many writings by Eringen [9–13], this balance law is believed to be not necessary, hence has never been used.

The work presented here establishes theoretical foundation for the necessity of the 'balance of moments of moments balance law' for solid continua using non-classical thermoelastic solids as an example. It is shown that in the absence of this balance law the constitutive theories become non-physical. Model problem studies are also presented to illustrate the necessity of this additional balance law in non-classical continuum theories for solids. The related literature review is presented in chapter 2.

The second investigation in this research considers derivation of the finite element formulations based on GM/WF for the mathematical models in non-classical continuum theories for the thermoelastic solid continua. These mathematical models when expressed purely in terms of displacements are merely balance of linear moment equations (for isothermal processes) that contain up to fourth order derivatives of displacements. Integral form resulting from GM/WF are shown to be variationally consistent, hence result in unconditionally stable computational processes. Integral form resulting from GM/WF are compared with those resulting in least squares processes (LSP) using a mathematical model consisting of first order PDEs. Meritorious and significant features of GM/WF over LSP are demonstrated and illustrated through model problem. Pertinent literature review related to this area of investigation are given in chapter 3. Chapter 4 contains some recommendations for future work.

Chapter 2

Necessity of balance of moments of moments balance law in non-classical continuum theories for solid continua

In non-classical continuum theories for solid continua the presence of internal rotations and their gradients arising due to Jacobian of deformation and consideration of Cosserat rotations [2] necessitate existence of moment tensor. For small deformation, small strains theories, in Lagrangian description the Cauchy moment tensor and the rates of rotation gradients are rate of work conjugate pair in addition to the rate of work conjugate Cauchy stress tensor and the strain rate tensor. It is well established that in such non-classical theories the Cauchy stress tensor is non-symmetric and the antisymmetric components of the Cauchy stress tensor are balanced by gradients of the Cauchy moment tensor, the balance of angular momenta balance law. *In the non-classical continuum theories incorporating internal rotations and conjugate moment tensor that are absent in the classical continuum theories, the fundamental question is "are the conservation and balance laws for classical continuum mechanics sufficient to ensure equilibrium of the deforming volume of matter"*. At this stage the Cauchy moment tensor remains non-symmetric if one only considers standard balance laws that are used in classical continuum theories. If one considers balance of moments of moments as a balance law, then the Cauchy moment tensor becomes symmetric, while in the absence of this balance law the Cauchy moment tensor remains non-symmetric. The purpose of this thesis is to establish theoretically as well as demonstrate through model problems that the balance of moments of moments is an essential balance law in non-classical continuum theories. Thermoelastic solid continua with small deformation and small strain is considered as an example for theoretical details as well as for the model problems. The conservation and balance laws as well as the constitutive theories are considered in the Lagrangian description. The findings reported in this thesis hold for thermoviscoelastic solids with and without memory as well when deformation and strains are small.

2.1 Literature review and scope of work

In classical continuum theories only the translations i.e displacements of the material points are considered as degrees of freedom in the conservation and balance laws. In case of small strains, small deformation of thermoelastic solids, the displacement gradient tensor $[{}^dJ]$ is the most fundamental measure of deformation, hence must form the basis for the derivation of conservation and balance laws in its entirely if complete deformation physics is to be included in resulting mathematical models. Surana et al. [3–5] have shown that polar decomposition of $[{}^dJ]$ into pure rotation and strech tensor or alternatively its decomposition into symmetric and skew-symmetric tensors separates strain measures and the rotation measures contained in $[{}^dJ]$. Thus, both strain measures as well as rotation measure must be considered in the derivations of the conservation and balance laws if $[{}^dJ]$ is to be incorporated in its entirety in the thermodynamic framework. Varying $[{}^dJ]$ between a material point and its neighbors when resisted by the deforming matter gives rise to conjugate moments. The moments and the rotations and their rates result in additional energy storage in thermoelastic solids. In thermoviscoelastic solids in addition to energy storage these also result in additional dissipation and rheology (memory). The internal rotations are completely defined by $[{}^dJ]$, hence are not unknown degrees of freedom at a material point. This physics due to internal rotations arising from $[{}^dJ]$ that exist in all deforming solids but is neglected in classical continuum theories. Similarly additional energy storage, dissipation and rheology also exists due to considerations of Cosserat rotations as unknown degrees of freedom at a material point in addition to internal rotations due to $[{}^dJ]$ [2]. The continuum theories that consider internal and/or Cosserat rotations in the thermodynamic framework are generally referred to as non-classical continuum theories.

Published account of non-classical theories dates back to Voigt [14, 15] in couple stress elasticity and subsequently the exhaustive work of Cosserat brothers [16]. A good literature review of the published work associated with non-classical continuum theories and its applications in solid mechanics and fluid mechanics can be found in recent paper by Surana et al. [2]. Significant aspects of the published works have been discussed in the references [3–47]. This material is omitted here, but the interested readers could refer to [2] or the original papers [3–47].

If the deforming solid matter is in thermodynamic equilibrium, then the conservation and balance laws must form the basis for admissible deformation. Thus, in classical continuum theories for continuous media, conservation of mass, balance of linear momenta, balance of angular momenta, first and second laws of thermodynamics must be satisfied. In the non-classical continuum theories for small deformation and small strain that incorporate internal rotations due to $[{}^dJ]$ and associate conjugate moments, the fundamental question of whether the conservation and balance laws derived for classical continuum theories indeed ensure equilibrium of the deforming solid matter in the presence of internal rotations and conjugate moments or is there a need for additional balance law due to the presence of new physics to ensure equilibrium is addressed in this chapter. Here one presents theoretical development as well as model problem studies for thermoelastic solids to demonstrate that in non-classical continuum theories that incorporate internal rotations due to $[{}^dJ]$ and associated conjugate moments, an additional balance law *"balance of moments of moments"* is necessary for the equilibrium of the deforming matter. The necessity of this balance law was first pointed out by Yang et al. [6] using geometric consideration. Since then this balance law has been used by Surana et al [3–5, 7, 8] successfully. It is shown that in absence of balance moments of moments balance law in the non-classical theories, when deriving the associated constitutive theories [2–5], inconsistencies are encountered in the derivation process for the constitutive theories [2–5]. The resulting constitutive theories when reduced to simple two dimensional deformation become nonphysical. Such inconsistencies and nonphysical deformation physics are completely avoided when *“balance of moments of moments”* is used as a balance law.

For the sake of simplicity one only consider thermoelastic solids with small deformation and small strains. Furthermore, only non-classical continuum theory that incorporates internal rotations due to $[{}^dJ]$ is considered. The material presented in chapter is also applicable for non-classical continuum theories that consider both internal and Cosserat rotation [2].

2.2 Notations, various measures, internal rotations and their gradients, and preliminary considerations

The notations used in this chapter follow reference [1] . Quantities with an over-bar are quantities in the current (deformed) configuration i.e. all quantities with over-bar are functions of coordinates $\bar{x}_i$ and time t, the Eulerian description. Quantities without an over-bar are quantities referred to the reference configuration i.e. these are functions of undeformed coordinates x_i and time t, Lagrangian description). The configuration at time $t = t_0 = 0$, commencement of evolution, is considered as the reference configuration. Thus, x_i and $\bar{x}_i$ are coordinates of the same material point in reference and current configurations, respectively, both measured in a fixed Cartesian x-frame. Here one only considers the Lagrangian description.

Consider the Jacobian of deformation defined by $\boldsymbol{J} = \boldsymbol{e}_i \otimes \boldsymbol{e}_j \frac{\partial \bar{x}_j}{\partial x_i}$. The rows are the covariant base vectors, whereas in Murnaghan's notation $[J] = \left[\frac{\partial\{\bar{x}\}}{\partial\{x\}}\right] = \left[\frac{\bar{x}_1,\bar{x}_2,\bar{x}_3}{x_1,x_2,x_3}\right]$, the columns are the covariant base vectors (i.e., in this definition $[J]$ is the transpose of $\boldsymbol{J}$ of the first definition). Both definitions are obviously covariant measures in the Lagrangian description. Likewise, $\bar{\boldsymbol{J}} = \boldsymbol{e}_i \otimes \boldsymbol{e}_j \frac{\partial x_j}{\partial \bar{x}_i}$ and $[\bar{J}] = \left[\frac{\partial\{x\}}{\partial\{\bar{x}\}}\right] = \left[\frac{x_1,x_2,x_3}{\bar{x}_1,\bar{x}_2,\bar{x}_3}\right]$ are also Jacobians of deformation but they are contravariant measures in the Eulerian description. Columns of $\bar{\boldsymbol{J}}$ are the contravariant base vectors whereas in case of $[\bar{J}]$ its rows are the contravariant base vectors (i.e., $\bar{\boldsymbol{J}}$ is transpose of $[\bar{J}]$). Here the corresponding Jacobians are denoted by $[J]$ and $[\bar{J}]$.

Since the work presented here only considers small strain and small rotations, the distinction between covariant and contravariant measures disappears as $\bar{x}_i \simeq x_i$ (i.e., the deformed configuration is not substantially different from the undeformed configuration). For such deformation, det$[J]$ = det$[\bar{J}] \cong 1$ and, hence, in the development of the theory there is a need to separate displacements from the deformed coordinates. The displacement gradient $[^dJ]$ is defined as

$$[^dJ] = \left[\frac{\partial\{u\}}{\partial\{x\}}\right] = \left[\frac{u_1, u_2, u_3}{x_1, x_2, x_3}\right] = \begin{bmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} & \dfrac{\partial u_1}{\partial x_3} \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & \dfrac{\partial u_2}{\partial x_3} \\ \dfrac{\partial u_3}{\partial x_1} & \dfrac{\partial u_3}{\partial x_2} & \dfrac{\partial u_3}{\partial x_3} \end{bmatrix} \tag{2.1}$$

The Cauchy stress tensor is used as a measure of stress because the deformed and undeformed tetrahedron can be treated the same for small deformation. Hence, the conservation and balance laws must be based entirety on $[^dJ]$ (i.e. $[^d_sJ]$ and $[^d_aJ]$ both must be considered in the conservation and balance laws).

The displacement gradient $[^dJ]$ can be written in component form as

$$^dJ_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) + \frac{1}{2}\left(u_{i,j} - u_{j,i}\right) = {}^d_sJ_{ij} + {}^d_aJ_{ij} \tag{2.2}$$

in which

$$[^d_aJ] = \begin{bmatrix} 0 & {}_i\Theta_{x_3} & -{}_i\Theta_{x_2} \\ -{}_i\Theta_{x_3} & 0 & {}_i\Theta_{x_1} \\ {}_i\Theta_{x_2} & -{}_i\Theta_{x_1} & 0 \end{bmatrix} \tag{2.3}$$

$${}_i\Theta_{x_1} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) \;;\quad {}_i\Theta_{x_2} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right) \;;\quad {}_i\Theta_{x_3} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) \tag{2.4}$$

Alternatively (2.4) can be derived as

$$\boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{e}_i \times \boldsymbol{e}_j \frac{\partial u_j}{\partial x_i} = \epsilon_{ijk} \boldsymbol{e}_k \frac{\partial u_j}{\partial x_i} \tag{2.5}$$

$$\boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{e}_1 \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \right) + \boldsymbol{e}_2 \left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \right) + \boldsymbol{e}_3 \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) \tag{2.6}$$

$$\boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{e}_1 \left(-2({}_i\Theta_{x_1}) \right) + \boldsymbol{e}_2 \left(-2({}_i\Theta_{x_2}) \right) + \boldsymbol{e}_3 \left(-2({}_i\Theta_{x_3}) \right) \tag{2.7}$$

The sign difference in (2.4) and (2.7) is due to the fact that rotations in (2.4) are in clockwise sense, whereas quantities in (2.6) are twice the magnitude compared to those in (2.4) and are in counterclockwise sense. Here (2.4) is considered as the definition of rotations (i.e., clockwise). The rotations defined in (2.4) exist at every material point in the deforming solid. $[{}^d_sJ]$ is a measure of infinitesimal strain where as $[{}^d_aJ]$ obviously contains rotation angle details. Gradients of the rotation angles can easily be obtained. Let

$$\{{}_i\Theta\}^T = [{}_i\Theta_{x_1},\ {}_i\Theta_{x_2},\ {}_i\Theta_{x_3}] \tag{2.8}$$

Gradients of rotations in (2.8) ($[{}^\Theta J]$) can be obtained using

$$\left[{}^\Theta J\right] = \left[\frac{\partial \{{}_i\Theta\}}{\partial \{x\}} \right] \quad \text{or} \quad {}^\Theta J_{ij} = \frac{\partial ({}_i\Theta_i)}{\partial x_j} \tag{2.9}$$

The gradient tensor $[{}^\Theta J]$ of rotations can be decomposed into symmetric and antisymmetric parts $[{}^\Theta_sJ]$ and $[{}^\Theta_aJ]$.

$$\left[{}^\Theta J\right] = \left[{}^\Theta_s J\right] + \left[{}^\Theta_a J\right] \tag{2.10}$$

$$\begin{aligned} \left[{}^\Theta_s J\right] &= \frac{1}{2} \left(\left[{}^\Theta J\right] + \left[{}^\Theta J\right]^T \right) \\ \left[{}^\Theta_a J\right] &= \frac{1}{2} \left(\left[{}^\Theta J\right] - \left[{}^\Theta J\right]^T \right) \end{aligned} \tag{2.11}$$

on the other hand polar decomposition of $[{}^dJ]$ gives

$$[{}^dJ] = [{}^dR][{}^dS_r] = [{}^dS_l][{}^dR] \tag{2.12}$$

The right and left stretch tensors $[{}^dS_r]$ and $[{}^dS_l]$ are symmetric and positive-definite, and $[{}^dR]$ is an orthogonal rotation tensor, a rotation matrix corresponding to the rotation angles defined in (2.4). $[{}^d_aJ]$ and $[{}^dR]$ contain the same physics as these both are derived from $[{}^dJ]$ but in different forms. $[{}^d_aJ]$ contains rotation angles while $[{}^dR]$ is the corresponding rotation matrix or tensor. Both in their forms given here can be used in derivations as needed. The same holds true for $[R]$ and $[{}_aJ]$ derived from $[J]$. However, deriving $[{}^dR]$ from $[{}^d_aJ]$ (or $[R]$ from $[{}_aJ]$) or vice versa in general in $\mathbb{R}^3$ may not be possible or unique [48–50]. Fortunately there is no need for this here.

Incorporating $[{}^dJ]$ in its entirety in the derivation of conservation and balance laws implies incorporating $[{}^d_sJ]$, and $[{}^d_aJ]$ (i.e., rotations ${}_i\Theta_{x_1}$, ${}_i\Theta_{x_2}$, and ${}_i\Theta_{x_3}$ about the axes of a triad located at each material point). Rotations in $[{}^d_aJ]$ are internal and are completely defined by skew-symmetric part of $[{}^dJ]$.

2.3 Considerations of stress, moment, and strain tensors

When the gradients of displacements vary between neighboring material points, so do the internal rotations ${}^d_a\boldsymbol{J}$. When the rotations ${}^d_a\boldsymbol{J}$ are resisted by the deforming matter conjugate moments are created. Rotations ${}^d_a\boldsymbol{J}$ and their rates and the conjugate moments can result in additional energy storage, dissipation, and rheology, i.e. in addition to those which are already present due to Cauchy stress tensor, strain, and strain rate tensors. Thus in the deforming solid matter rotations ${}^d_a\boldsymbol{J}$ that are conjugate to moment tensor necessitates that on the boundary of the deformed volume there must exist a resultant moment.

Consider a volume of matter $\underset{\sim}{V}$ in the reference configuration with closed boundary $\partial\underset{\sim}{V}$. The volume V is isolated from $\underset{\sim}{V}$ by a hypothetical surface ∂V as in the cut principle of Cauchy. Consider a tetrahedron T_1 such that its oblique plane is part of ∂V and its other three planes are orthogonal to each other and parallel to the planes of the x-frame. Upon deformation, $\underset{\sim}{V}$ and $\partial\underset{\sim}{V}$

occupy $\bar{\underset{\sim}{V}}$ and $\partial\bar{\underset{\sim}{V}}$ and likewise V and ∂V deform into $\bar{V}$ and $\partial\bar{V}$. The tetrahedron T_1 deforms into $\bar{T}_1$ whose edges (under finite deformation) are non-orthogonal covariant base vectors $\tilde{g}_i$. The planes of the tetrahedron formed by the covariant base vectors are flat but obviously non-orthogonal to each other. Tetrahedron is assumed to be in the small neighborhood of material point $\bar{o}$ so that the assumption of the oblique plane $\bar{A}\bar{B}\bar{C}$ being flat but still part of $\partial\bar{V}$ is valid. When the deformed tetrahedron is isolated from volume $\bar{V}$ it must be in equilibrium under the action of disturbance on surface $\bar{A}\bar{B}\bar{C}$ from the volume surrounding $\bar{V}$ and the internal fields that act on the flat faces which equilibrate with the mating faces in volume $\bar{V}$ when the tetrahedron T_2 is placed back in the volume $\bar{V}$.

Consider the deformed tetrahedron $\bar{T}_1$. Let $\bar{\boldsymbol{P}}$ be the average stress per unit area on plane $\bar{A}\bar{B}\bar{C}$, $\bar{\boldsymbol{M}}$ be the average moment per unit area on plane $\bar{A}\bar{B}\bar{C}$ henceforth referred to as moment for short, and $\bar{\boldsymbol{n}}$ be the normal to the face $\bar{A}\bar{B}\bar{C}$. $\bar{\boldsymbol{P}}$, $\bar{\boldsymbol{M}}$, and $\bar{\boldsymbol{n}}$ all have different directions when the deformation is finite. Based on the small deformation assumption, the deformed coordinates $\bar{x}_i$ are approximately same as undeformed coordinates x_i, thus the deformed tetrahedron $\bar{T}_1$ in the current configuration is close to its map T_1 in the reference configuration. With this assumption all stress measures (first and second Piola-Kirchhoff stress tensors, Cauchy stress tensor) are approximately the same. The same holds for the moment tensors. Thus with the assumption $\bar{\boldsymbol{x}} \simeq \boldsymbol{x}$ can be written as

$$\bar{\boldsymbol{P}} = \boldsymbol{P} \ , \quad \bar{\boldsymbol{M}} = \boldsymbol{M} \tag{2.13}$$

The Cauchy principle for $\bar{\boldsymbol{P}}$ and $\bar{\boldsymbol{M}}$ gives (hence for $\boldsymbol{P}$ and $\boldsymbol{M}$)

$$\boldsymbol{P} = \boldsymbol{\sigma} \cdot \boldsymbol{n} \ , \quad \boldsymbol{M} = \boldsymbol{m} \cdot \boldsymbol{n} \tag{2.14}$$

in which $\boldsymbol{\sigma}$ is Cauchy stress tensor and $\boldsymbol{m}$ is Cauchy moment tensor (per unit area), both non-symmetric at this stage. Since one only considers small deformation small strain, symmetric part of ${}^{d}\boldsymbol{J}$ i.e ${}^{d}_{s}\boldsymbol{J}$ is in fact the strain tensor.

$$\boldsymbol{\varepsilon} = {}^{d}_{s}\boldsymbol{J} \tag{2.15}$$

2.4 Conservation and balance laws and the constitutive theories for thermoelastic solid in the absence of balance of moments of moments as a balance law

First one considers conservation and balance laws for deforming solids with internal rotation physics assuming that balance of moments of moments is not a required balance law. This requires consideration of standard conservation and balance laws: conservation of mass, balance of linear momenta, balance of angular momenta, first and second laws of thermodynamics and their modifications to incorporate physics of deformation due to ${}^{d}_{a}\boldsymbol{J}$, the internal rotations. This is then followed by the consideration of the second law of thermodynamics and conditions resulting from it that are then used in deriving consistent set of constitutive theories that satisfy the conditions resulting from the second law of thermodynamics.

2.4.1 Conservation of mass and balance of linear momenta

The continuity equation resulting from the principle of conservation of mass remains the same for the non-classical continuum theory considered here as in case of classical continuum theory as long as the matter is treated homogeneous and isotropic. In Lagrangian description, continuity equation [1, 51] can be written as

$$\rho_0(\boldsymbol{x}) = |J|\rho(\boldsymbol{x}, t) \tag{2.16}$$

For infinitesimal deformation $|J| \simeq 1$ hence,

$$\rho_0(\boldsymbol{x}) = \rho(\boldsymbol{x}, t) \tag{2.17}$$

where $\rho_0(\boldsymbol{x})$ is the density of the material point at $\boldsymbol{x}$ in the reference configuration and $\rho(\boldsymbol{x}, t)$ is the Lagrangian description of the density of a material point at $\bar{\boldsymbol{x}}$ in the current configuration.

For a deforming volume of matter the rate of change of linear momenta must be equal to the sum of all other forces acting on it. This is Newton's second law applied to a volume of matter. The derivation is same as that for classical continuum theory. Thus, one can write (for small deformation) the following [1, 51]:

$$\rho_0 \frac{D\boldsymbol{v}}{Dt} - \rho_0 \boldsymbol{F}^b - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = 0$$

or (2.18)

$$\rho_0 \frac{D\{v\}}{Dt} - \rho_0 \{F^b\} - [\sigma]^T \{\nabla\} = 0$$

In Lagrangian description $\frac{D}{Dt} = \frac{\partial}{\partial t}$ and $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x}, t)$ are velocities, $\boldsymbol{F}^b$ are body forces per unit mass, and $\boldsymbol{\sigma}$ is the Cauchy stress tensor. Equations (2.18) are momentum equations in x_1-, x_2-, and x_3-directions. The Cauchy stress tensor is nonsymmetric at this stage as its symmetry has not been established.

2.4.2 Balance of angular momenta

Balance of angular momenta for deforming solids with internal rotation physics has been derived by Surana et al. [3, 4], but is presented in the following as this represents the first significant deviation from the classical continuum theories in which this balance law simply establishes that the Cauchy stress tensor is symmetric.

The principle of balance of angular momenta for a non-classical continuum with internal rotations in present case can be stated as follows: *The time rate of change of total moment of momenta for a non-classical continuum is equal to the vector sum of the moments of external forces and the moments*. Thus, due to the surface stress $\bar{\boldsymbol{P}}$, total surface moment $\bar{\boldsymbol{M}}$ (per unit area) created when the internal rotations are resisted by the deforming continuum, body force $\bar{\boldsymbol{F}}^b$ (per unit mass), and the momentum $\bar{\rho}\bar{\boldsymbol{v}}d\bar{V}$ for an elemental mass $\bar{\rho}d\bar{V}$ in the current configuration (using the Eulerian description) one can write the following:

$$\frac{D}{Dt}\int\limits_{\bar{V}(t)} \bar{\boldsymbol{x}} \times \bar{\rho}\bar{\boldsymbol{v}}\, d\bar{V} = \int\limits_{\partial\bar{V}(t)} \left(\bar{\boldsymbol{x}} \times \bar{\boldsymbol{P}} - \bar{\boldsymbol{M}}\right)\, d\bar{A} + \int\limits_{\bar{V}(t)} \bar{\boldsymbol{x}} \times \bar{\rho}\bar{\boldsymbol{F}}^{b}\, d\bar{V} \tag{2.19}$$

Negative sign for $\bar{\boldsymbol{M}}$ is due to clockwise rotations being positive. One considers each term in (2.19) individually. First consider

$$\begin{aligned}\frac{D}{Dt}\int\limits_{\bar{V}(t)} \bar{\boldsymbol{x}} \times \bar{\rho}\bar{\boldsymbol{v}} d\bar{V} &= \frac{D}{Dt}\int\limits_{\bar{V}(t)} \epsilon_{ijk}\bar{x}_i\bar{v}_j\boldsymbol{e}_k\bar{\rho}\, d\bar{V} = \frac{D}{Dt}\int\limits_{V} \epsilon_{ijk}\boldsymbol{e}_k x_i v_j \rho_0\, dV \\ &= \int\limits_{V} \rho_0 \epsilon_{ijk}\boldsymbol{e}_k \frac{D}{Dt}\left(x_i v_j\right) dV = \int\limits_{V} \rho_0\epsilon_{ijk}\boldsymbol{e}_k \left(v_i v_j + x_i\frac{Dv_j}{Dt}\right) dV \\ &= \int\limits_{V} \rho_0 \epsilon_{ijk}\boldsymbol{e}_k x_i \frac{Dv_j}{Dt}\, dV\end{aligned} \tag{2.20}$$

Consider the first term on the right hand side of (2.19):

$$\begin{aligned}\int\limits_{\partial\bar{V}(t)} \left(\bar{\boldsymbol{x}} \times \bar{\boldsymbol{P}} - \bar{\boldsymbol{M}}\right) d\bar{A} &= \int\limits_{\partial\bar{V}(t)} \left[\bar{\boldsymbol{x}} \times (\bar{\boldsymbol{\sigma}})^T \cdot \bar{\boldsymbol{n}} - (\bar{\boldsymbol{m}})^T \cdot \bar{\boldsymbol{n}}\right] d\bar{A} \\ &= \int\limits_{\partial V} \left[\boldsymbol{x} \times (\boldsymbol{\sigma})^T \cdot \boldsymbol{n} - (\boldsymbol{m})^T \cdot \boldsymbol{n}\right] dA \\ &= \int\limits_{\partial V} \boldsymbol{e}_k \left(\epsilon_{ijk} x_i \sigma_{mj} n_m - m_{mk} n_m\right) dA \\ &= \int\limits_{V} \boldsymbol{e}_k \left[\epsilon_{ijk}\left(x_i\sigma_{mj}\right)_{,m} - m_{mk,m}\right] dV \\ &= \int\limits_{V} \boldsymbol{e}_k \left[\epsilon_{ijk}\left(\sigma_{ij} + x_i(\sigma_{mj})_{,m}\right) - m_{mk,m}\right] dV\end{aligned} \tag{2.21}$$

in which $\bar{\boldsymbol{\sigma}}$ is the *Cauchy stress tensor* and $\bar{\boldsymbol{m}}$ is the *Cauchy moment tensor*. Next, consider the second term on the right hand side (2.19):

$$\int\limits_{\bar{V}(t)} \bar{\boldsymbol{x}} \times \bar{\rho}\bar{\boldsymbol{F}}^{b} d\bar{V} = \int\limits_{\bar{V}(t)} \boldsymbol{e}_k \epsilon_{ijk}\bar{x}_i \bar{F}_j^b \bar{\rho} d\bar{V} = \int\limits_{V} \boldsymbol{e}_k\epsilon_{ijk} x_i {F_j}^b \rho_0 dV \tag{2.22}$$

Substituting from (2.20), (2.21), and (2.22) into (2.19), one obtains

$$\int_V \boldsymbol{e}_k \epsilon_{ijk} \left[x_i \left(\rho_0 \frac{Dv_j}{Dt} - \rho_0 F_j^b - \sigma_{mj,m} \right) \right] dV + \int_V \boldsymbol{e}_k (m_{mk,m} - \epsilon_{ijk}\sigma_{ij})\, dV = 0 \tag{2.23}$$

Using balance of linear momenta (2.18) in (2.23), one arrives at

$$\int_V \boldsymbol{e}_k \left(m_{mk,m} - \epsilon_{ijk}\sigma_{ij} \right) dV = 0 \tag{2.24}$$

and, since the volume V is arbitrary:

$$m_{mk,m} - \epsilon_{ijk}\sigma_{ij} = 0 \tag{2.25}$$

$$\text{or} \qquad \boldsymbol{\nabla} \cdot \boldsymbol{m} - \boldsymbol{\epsilon} : \boldsymbol{\sigma} = 0 \tag{2.26}$$

Equation (2.25) represents balance of angular momenta. *The Cauchy stress tensor $\boldsymbol{\sigma}$ and moment stress tensor $\boldsymbol{m}$ is non-symmetric*. From (2.25) one notes that *antisymmetric components of the Cauchy stress tensor $\boldsymbol{\sigma}$ are balanced by the gradients of the Cauchy moment tensor*.

Remarks.

(a) In the balance of angular momenta, the rate of change of angular momenta is balanced by the vector sum of the moments of the forces. Thus, this balance law naturally contains moments due to components of the stress tensor acting on the faces of the deformed tetrahedron. Normal stress components do not contribute to this. Hence, the moments contained in this balance law due to stresses are only caused by the shear stresses contained in the skew-symmetric part of the Cauchy stress tensor.

(b) In the case of classical continuum theory, the balance of angular momenta is a statement of self equilibrating moments due to the symmetry of shear stresses

$$\boldsymbol{\varepsilon} : \boldsymbol{\sigma} = 0 \tag{2.27}$$

An important point to note is that (2.27) is a result of stress couples due to shear stresses.

(c) In the case of non-classical continua, the existence of moments $\boldsymbol{m}$ due to the material constitution resisting the internal rotations, results in the shear stress couples from the antisymmetric part of the Cauchy stress tensor being balanced by the internal moments. Thus, for non-classical continua, the balance of angular momenta yields (2.26) instead of (2.27).

(d) Both the non-classical and classical continuum theories use stress couples in the balance of angular momenta.

(e) From (2.25) it is clear that gradients of $\boldsymbol{m}$ equilibrate only with the antisymmetric (shear) components of the stress tensor $\boldsymbol{\sigma}$.

(f) Varying internal rotations at the neighboring material points when resisted by the deforming matter require existence of internal moment tensor $\boldsymbol{m}$. The balance of angular momenta establishes a relationship between $\boldsymbol{m}$ and $\boldsymbol{\sigma}$.

2.4.3 First law of thermodynamics

The sum of work and heat added to a deforming volume of matter must result in increase of the energy of the volume of matter. This is expressed as a rate equation in the Eulerian description as

$$\frac{D\bar{E}_t}{Dt} = \frac{D\bar{Q}}{Dt} + \frac{D\bar{W}}{Dt} \tag{2.28}$$

where $\bar{E}_t$, $\bar{Q}$ and $\bar{W}$ are total energy, heat added, and work done. One notes

$$\frac{D\bar{E}_t}{Dt} = \frac{D}{Dt} \int\limits_{\bar{V}(t)} \bar{\rho} \left(\bar{e} + \frac{1}{2} \bar{\boldsymbol{v}} \cdot \bar{\boldsymbol{v}} - \bar{\boldsymbol{F}}^b \cdot \bar{\boldsymbol{u}} \right) d\bar{V} \tag{2.29}$$

$$\frac{D\bar{Q}}{Dt} = - \int\limits_{\partial \bar{V}(t)} \bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{n}} \, d\bar{A} \tag{2.30}$$

$$\frac{D\bar{W}}{Dt} = \int\limits_{\partial\bar{V}(t)} (\bar{\boldsymbol{P}} \cdot \bar{\boldsymbol{v}} + \bar{\boldsymbol{M}} \cdot {}_i\bar{\boldsymbol{\Theta}})\, d\bar{A} \tag{2.31}$$

in which ${}_i\bar{\boldsymbol{\Theta}}$ are rotation rates in Eulerian description. Using (2.29)-(2.31) in (2.28) and following the derivation presented by Surana et al. [1] one can derive the following energy equation in Lagrangian description.

$$\rho_0 \frac{De}{Dt} + \boldsymbol{\nabla} \cdot \boldsymbol{q} - \sigma_{ji} \frac{\partial v_i}{\partial x_j} - \left(m_{ji} \frac{\partial ({}_i\dot{\Theta}_i)}{\partial x_j} + {}_i\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{m}) \right) = 0 \tag{2.32}$$

One notes that in ${}_i\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{m})$, the term $\boldsymbol{\nabla} \cdot \boldsymbol{m}$ can be substituted from (2.26) thereby eliminating gradients of $\boldsymbol{m}$ but introducing $\boldsymbol{\sigma}$ in its place.

One notes that

$$\begin{aligned} \frac{\partial v_i}{\partial x_j} &= {}^d\dot{J}_{ij} \\ \text{hence,} \qquad \sigma_{ji} \frac{\partial v_i}{\partial x_j} &= \sigma_{ji}\, {}^d\dot{J}_{ij} = \text{tr}([\sigma][{}^d\dot{J}]) \end{aligned} \tag{2.33}$$

and

$$\begin{aligned} \frac{\partial ({}_i\dot{\Theta}_i)}{\partial x_j} &= {}^\Theta\dot{J}_{ij} \\ \text{Therfore,} \qquad m_{ji} \frac{\partial ({}_i\dot{\Theta}_i)}{\partial x_j} &= \text{tr}([m][{}^\Theta\dot{J}]) \end{aligned} \tag{2.34}$$

Hence, (2.32) can be written as

$$\rho_0 \frac{De}{Dt} + \boldsymbol{\nabla} \cdot \boldsymbol{q} - \text{tr}([\sigma][{}^d\dot{J}]) - \Big(\text{tr}([m][{}^\Theta\dot{J}]) + {}_i\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{m}) \Big) = 0 \tag{2.35}$$

2.4.4 Second law of thermodynamics

Derivation of the entropy inequality using second law of thermodynamics follows standard approach used in classical continuum theories upto the derivation of Clausius Duham inequality given by

$$\rho_0 \theta \frac{D\eta}{Dt} + \left(\boldsymbol{\nabla} \boldsymbol{\cdot} \boldsymbol{q} - \rho_0 r\right) - \frac{1}{\theta} \boldsymbol{q} \boldsymbol{\cdot} \boldsymbol{g} \geq 0 \tag{2.36}$$

In (2.36), $\left(\boldsymbol{\nabla} \boldsymbol{\cdot} \boldsymbol{q} - \rho_0 r\right)$ are substituted from the energy equation, then Helmholtz free energy density Φ is introduced using

$$\Phi = e - \eta\theta \tag{2.37}$$

to finally obtain

$$\rho_0 \left(\frac{D\Phi}{Dt} + \eta \frac{D\theta}{Dt} \right) + \frac{\boldsymbol{q} \boldsymbol{\cdot} \boldsymbol{g}}{\theta} - \text{tr}\left([\sigma][{}^{d}\dot{J}] \right) - \text{tr}\left([m][{}^{\Theta}\dot{J}] \right) - {}_{i}\dot{\boldsymbol{\Theta}} \boldsymbol{\cdot} (\boldsymbol{\nabla} \boldsymbol{\cdot} \boldsymbol{m}) \leq 0 \tag{2.38}$$

Equation (2.38) is the final form of the second law of thermodynamics (entropy inequality) at this point for non-classical continuum theory incorporating internal rotations.

2.4.5 Constitutive theories for thermoelastic solids in the absence of balance of moments of moments balance law

For thermoelastic solids, the mechanical deformation is reversible, that is, the rate of work gets stored as rate of strain energy density and there is no rate of entropy production due to the mechanical deformation process. In such solids, the constitutive theories (for $\boldsymbol{\sigma}$ and $\boldsymbol{m}$ as shown later) can be derived using Helmholtz free energy density Φ provided its argument tensors and the conjugate pairs in the constitutive theories are known. From (2.38) it appears that $([\sigma], [{}^{d}\dot{J}])$ and $([m], [{}^{\Theta}\dot{J}])$ are rate of work conjugate pairs. However, the additional term ${}_{i}\dot{\boldsymbol{\Theta}} \boldsymbol{\cdot} (\boldsymbol{\epsilon} : \boldsymbol{\sigma})$ also needs to be accounted for. Details of all these considerations are presented in the following. The simplest possible constitutive theory for $\boldsymbol{q}$ can be derived using $\boldsymbol{q} \boldsymbol{\cdot} \boldsymbol{g} \leq 0$ (shown later).

Once the rate of work conjugate pairs are established from the energy equation or entropy inequality, the constitutive theories can also be derived using these in conjunction with the representation theorem or theory of generators and invariants. Details are considered in the following. One notes that $[\sigma]$, $[{}^{d}\dot{J}]$ are both non-symmetric tensors and so are $[m]$, $[{}^{\Theta}\dot{J}]$. Whether $([\sigma], [{}^{d}\dot{J}])$

and ($[m]$, $[{}^{\Theta}\dot{J}]$) are rate of work conjugate pairs or not needs to be established.

2.4.5.1 Rate of work conjugate pairs in the entropy inequality

Based on works of Spencer,Wang, Zheng, etc. [52–68], one notes that constitutive theories for $[\sigma]$, $[m]$ cannot be derived in terms of $[{}^{d}J]$ and $[{}^{\Theta}J]$, as these are all non-symmetric tensors. Based on representation theorem [1,51–68], a non-symmetric tensor cannot have another non-symmetric tensor as its argument tensor as in this case integrity cannot be established using the argument tensor i.e,

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}({}^{d}\dot{J}); \quad \text{and,} \quad \boldsymbol{m} = \boldsymbol{m}({}^{\Theta}\dot{J}) \tag{2.39}$$

does not hold. Thus, at this stage ($[\sigma]$, $[{}^{d}\dot{J}]$) and ($[m]$, $[{}^{\Theta}\dot{J}]$) are not rate of work conjugate pairs.

2.4.5.2 Rate of work conjugate pairs

Decomposing $[\sigma]$,$[m]$,$[\dot{J}]$ and $[{}^{\Theta}\dot{J}]$ into symmetric and antisymmetric tensors.

$$\begin{aligned}
\boldsymbol{\sigma} &= {}_{s}\boldsymbol{\sigma} + {}_{a}\boldsymbol{\sigma} \\
\boldsymbol{m} &= {}_{s}\boldsymbol{m} + {}_{a}\boldsymbol{m} \\
{}^{d}\dot{\boldsymbol{J}} &= {}^{d}_{s}\dot{\boldsymbol{J}} + {}^{d}_{a}\dot{\boldsymbol{J}} \\
{}^{\Theta}\dot{\boldsymbol{J}} &= {}^{\Theta}_{s}\dot{\boldsymbol{J}} + {}^{\Theta}_{a}\dot{\boldsymbol{J}}
\end{aligned} \tag{2.40}$$

Substituting (2.40) into (2.38) and noting that

$$\begin{aligned}
\text{tr}\left([{}_{s}\sigma][{}^{d}_{a}\dot{J}]\right) &= 0 \\
\text{tr}\left([{}_{a}\sigma][{}^{d}_{s}\dot{J}]\right) &= 0 \\
\text{tr}\left([{}_{s}m][{}^{\Theta}_{a}\dot{J}]\right) &= 0 \\
\text{tr}\left([{}_{a}m][{}^{\Theta}_{s}\dot{J}]\right) &= 0
\end{aligned} \tag{2.41}$$

and

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \boldsymbol{m} &= \boldsymbol{\epsilon} : \boldsymbol{\sigma} \\
{}_i\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{m}) &= {}_i\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\epsilon} : \boldsymbol{\sigma}) \\
\text{tr}\left([{}_a\sigma][{}^d_a\dot{J}]\right) &= -{}_i\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\epsilon} : \boldsymbol{\sigma})
\end{aligned}
\tag{2.42}
$$

Using (2.40),(2.41),(2.42), one can write (2.38) as

$$
\rho_0 \left(\frac{D\Phi}{Dt} + \eta \frac{D\theta}{Dt}\right) + \frac{\boldsymbol{q} \cdot \boldsymbol{g}}{\theta} - \text{tr}\left([{}_s\sigma][{}^d_s\dot{J}]\right) - \text{tr}\left([{}_s m][{}^\Theta_s\dot{J}]\right) - \text{tr}\left([{}_a m][{}^\Theta_a\dot{J}]\right) \leq 0 \tag{2.43}
$$

The energy equation is accordingly modified as well and one can write

$$
\rho_0 \frac{De}{Dt} + \boldsymbol{\nabla} \cdot \boldsymbol{q} - \text{tr}\left([{}_s\sigma][{}^d_s\dot{J}]\right) - \text{tr}\left([{}_s m][{}^\Theta_s\dot{J}]\right) - \text{tr}\left([{}_a m][{}^\Theta_a\dot{J}]\right) = 0 \tag{2.44}
$$

From entropy inequality (2.43), one can conclude that $\left([{}_s\sigma], [{}^d_s\dot{J}]\right)$;$\left([{}_s m], [{}^\Theta_s\dot{J}]\right)$;$\left([{}_a m], [{}^\Theta_a\dot{J}]\right)$ are rate of work conjugate pairs. In these conjugate pairs, symmetric tensors are conjugate to symmetric tensors and antisymmetric tensor is conjugate to antisymmetric tensor. These are in conformity with the representation theorem and the works of Spencer, Wang, Zheng, etc. [52–68]. This mathematical model has dependent variables: $v(3)$, ${}_s\boldsymbol{\sigma}(6)$, ${}_a\boldsymbol{\sigma}(3)$, ${}_s\boldsymbol{m}(6)$, ${}_a\boldsymbol{m}(3)$, $\boldsymbol{q}(3)$, $\theta(1)$: a total of 25. Φ, η and e are not dependent variables as these can be expressed in terms of others. The members in the bracket refers to number of variables. The equations in the model are: linear momenta (3), angular momenta (3), energy equation(1), constitutive theories for ${}_s\boldsymbol{\sigma}(6)$, ${}_s\boldsymbol{m}(6)$, ${}_a\boldsymbol{m}(3)$, $\boldsymbol{q}(3)$, a total of 25, hence mathematical model has closure.

2.4.5.3 Dependent variables in the constitutive theories and their argument tensor

Choice of Φ, η, ${}_s\boldsymbol{\sigma}$, ${}_s\boldsymbol{m}$, ${}_a\boldsymbol{m}$, and $\boldsymbol{q}$ as dependent variables in the constitutive theories is rather straight forward from the balance laws (more specifically entropy inequality). Their argument tensors are dictated by the conjugate pairs in the entropy inequality and the fact that for thermoelastic solids mechanical deformation is reversible, hence the Helmholtz energy density Φ must contain mechanism for energy storage. Based on these considerations one can write (for the simplest case).

$$\begin{aligned}
\Phi &= \Phi({}^{\boldsymbol{d}}_{\boldsymbol{s}}\boldsymbol{J}, {}^{\boldsymbol{\Theta}}_{\boldsymbol{s}}\boldsymbol{J}, {}^{\boldsymbol{\Theta}}_{\boldsymbol{a}}\boldsymbol{J}, \boldsymbol{g}, \theta) \\
\eta &= \eta({}^{\boldsymbol{d}}_{\boldsymbol{s}}\boldsymbol{J}, {}^{\boldsymbol{\Theta}}_{\boldsymbol{s}}\boldsymbol{J}, {}^{\boldsymbol{\Theta}}_{\boldsymbol{a}}\boldsymbol{J}, \boldsymbol{g}, \theta) \\
{}_{\boldsymbol{s}}\boldsymbol{\sigma} &= {}_{\boldsymbol{s}}\boldsymbol{\sigma}({}^{\boldsymbol{d}}_{\boldsymbol{s}}\boldsymbol{J}, \theta) = {}_{s}\boldsymbol{\sigma}(\boldsymbol{\varepsilon}, \theta); \quad as \quad {}^{\boldsymbol{d}}_{\boldsymbol{s}}\boldsymbol{J} = \boldsymbol{\varepsilon}, \text{linear strain} \\
{}_{\boldsymbol{s}}\boldsymbol{m} &= {}_{\boldsymbol{s}}\boldsymbol{m}({}^{\boldsymbol{\Theta}}_{\boldsymbol{s}}\boldsymbol{J}, \theta) \\
{}_{\boldsymbol{a}}\boldsymbol{m} &= {}_{\boldsymbol{a}}\boldsymbol{m}({}^{\boldsymbol{\Theta}}_{\boldsymbol{a}}\boldsymbol{J}, \theta) \\
\boldsymbol{q} &= \boldsymbol{q}(\boldsymbol{g}, \theta)
\end{aligned} \tag{2.45}$$

Since the argument tensor of Φ in (2.45) is known, one can obtain $\frac{D\Phi}{Dt} = \dot{\Phi}$ by using chain rule of differentiation and then substitute it in entropy inequality. After regrouping terms, it is found that entropy inequality is satisfied for arbitrary but admissible ${}^{d}_{s}\dot{J}$, ${}^{\Theta}_{s}\dot{J}$, ${}^{\Theta}_{a}\dot{J}$, $\dot{g}$ and $\dot{\theta}$ if the following hold

$$ {}_{s}\sigma_{ki} = \rho_0 \frac{\partial \Phi}{\partial({}^{d}_{s}J_{ki})} \tag{2.46}$$

$$ {}_{s}m_{ki} = \rho_0 \frac{\partial \Phi}{\partial({}^{\Theta}_{s}J_{ki})} \tag{2.47}$$

$$ {}_{a}m_{ki} = \rho_0 \frac{\partial \Phi}{\partial({}^{\Theta}_{a}J_{ki})} \tag{2.48}$$

$$ \eta = -\frac{\partial \Phi}{\partial \theta}; \tag{2.49}$$

$$ \Phi \neq \Phi(\boldsymbol{g});\ \Phi \text{ is not a funtion of } \boldsymbol{g} \tag{2.50}$$

when (2.46)-(2.50) hold, the entropy inequality ((2.43)) reduces to

$$ \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} \leq 0 \tag{2.51}$$

Equation (2.49) implies that η is not a dependent variable in the constitutive theories.

Remarks.

(1) Constitutive theories for ${}_s\boldsymbol{\sigma}$, ${}_s\boldsymbol{m}$, ${}_a\boldsymbol{m}$, can be derived using (2.46)-(2.48) and assuming that Φ is a function of the invariants of $\boldsymbol{\varepsilon}$, $\overset{\Theta}{\underset{s}{}}\boldsymbol{J}$ and $\overset{\Theta}{\underset{a}{}}\boldsymbol{J}$ (due to frame invariance requirement)

(2) Constitutive theories for ${}_s\boldsymbol{\sigma}$, ${}_s\boldsymbol{m}$ and ${}_a\boldsymbol{m}$ can also be derived using their argument tensors defined in (2.45) and the representation theorems.

(3) Constitutive theory for $\boldsymbol{q}$ can be derived using (2.47) or the argument tensors of $\boldsymbol{q}$ in (2.45) and the representation theorem.

(4) As usual, the material coefficients in each constitutive theory are established using Taylor series expansion of the coefficients in the linear combination in terms of invariant and θ about a known configuration.

Details are presented in the following.

2.4.5.4 Constitutive theory for ${}_s\boldsymbol{\sigma}$

Constitutive theories for ${}_s\boldsymbol{\sigma}$ can be derived using (2.46) and by considering $\Phi = \Phi(I_\varepsilon, II_\varepsilon, III_\varepsilon, \theta)$ or by considering ${}_s\boldsymbol{\sigma} = {}_s\boldsymbol{\sigma}(\boldsymbol{\varepsilon}, \theta)$ and using representation theorem. Both approaches are considered.

2.4.5.4.1 Constitutive theory for ${}_s\boldsymbol{\sigma}$ using Φ as a function of the invariants of $\boldsymbol{\varepsilon}$ and θ

Constitutive theory for ${}_s\boldsymbol{\sigma}$ is derived using Φ as a function of the invariants of $\boldsymbol{\varepsilon}$ and θ. Using

$$ {}_s\boldsymbol{\sigma} = \rho_0 \frac{\partial \Phi(I_\varepsilon, II_\varepsilon, III_\varepsilon, \theta)}{\partial \boldsymbol{\varepsilon}} \tag{2.52} $$

$I_\varepsilon, II_\varepsilon, III_\varepsilon$ are principal invariants of ε a symmetric tensor of rank 2. Using (2.52) it is straightforward to derive the following for ${}_s\boldsymbol{\sigma}$

$$
{}_s\boldsymbol{\sigma} = {}^{\sigma}\alpha^0 \boldsymbol{I} + {}^{\sigma}\alpha^1 \boldsymbol{\varepsilon} + {}^{\sigma}\alpha^2 \boldsymbol{\varepsilon}^{-1} \tag{2.53}
$$

in which

$$
\begin{aligned}
{}^{\sigma}\alpha^0 &= \rho_0 \left(\frac{\partial \Phi}{\partial II_\varepsilon} + \frac{\partial \Phi}{\partial III_\varepsilon} I_\varepsilon \right) \\
{}^{\sigma}\alpha^1 &= -\rho_0 \frac{\partial \Phi}{\partial II_\varepsilon} \\
{}^{\sigma}\alpha^2 &= \rho_0 \frac{\partial \Phi}{\partial III_\varepsilon}
\end{aligned} \tag{2.54}
$$

Using the Cayley-Hamilton theorem [1], (2.53) can be written as

$$
{}_s\boldsymbol{\sigma} = {}^{\sigma}\tilde{\alpha}^0 \boldsymbol{I} + {}^{\sigma}\tilde{\alpha}^1 \boldsymbol{\varepsilon} + {}^{\sigma}\tilde{\alpha}^2 \boldsymbol{\varepsilon}^{2} \tag{2.55}
$$

in which ${}^{\sigma}\tilde{\alpha}^i$; $i = 0, 1, 2$ are functions of I_ε, II_ε, and III_ε, and θ in the current configuration. Material coefficients are derived using ${}^{\sigma}\tilde{\alpha}$= ${}^{\sigma}\tilde{\alpha}^i(I_\varepsilon, II_\varepsilon, III_\varepsilon, \theta)$; $i = 0, 1, 2$.

2.4.5.4.2 Constitutive theory for ${}_s\boldsymbol{\sigma}$ using representation theorem

Consider

$$
{}_s\boldsymbol{\sigma} = {}_s\boldsymbol{\sigma}(\boldsymbol{\varepsilon}, \theta) \tag{2.56}
$$

${}_s\boldsymbol{\sigma}$ is a symmetric tensor of rank two whose arguments are $\boldsymbol{\varepsilon}$, a symmetric tensor of rank two, and θ, a tensor of rank zero. Based on representation theorem [1, 51–68], ${}_s\boldsymbol{\sigma}$ can be expressed as a linear combination of the combined generators of its arguments that are symmetric tensors of rank two. The combined generators of $\boldsymbol{\varepsilon}$ and θ that are symmetric tensors of rank two are [1] $\boldsymbol{I}$, $\boldsymbol{\varepsilon}$, and

$\boldsymbol{\varepsilon}^2$. Using the same coefficients as in (2.55) one can write

$$ {}_s\boldsymbol{\sigma} = {}^\sigma\tilde{\alpha}^0 \boldsymbol{I} + {}^\sigma\tilde{\alpha}^1 \boldsymbol{\varepsilon} + {}^\sigma\tilde{\alpha}^2 \boldsymbol{\varepsilon}^2 \tag{2.57} $$

Here also ${}^\sigma\tilde{\alpha}^i = {}^\sigma\tilde{\alpha}^i(I_\varepsilon, II_\varepsilon, III_\varepsilon, \theta)$; $i = 0, 1, 2$. Constitutive theory (2.57) is clearly same as (2.55). Invariants i_ε, $\ddot{u}_\varepsilon$, and $\dddot{w}_\varepsilon$ can also be used instead of I_ε, II_ε, and III_ε as the two sets of invariants are related. The final outcome remains unaffected.

2.4.5.4.3 Material coefficients in the constitutive theory for ${}_s\boldsymbol{\sigma}$

To derive the material coefficients for the constitutive theory for ${}_s\boldsymbol{\sigma}$ in (2.53) or (2.57), we expand ${}^\sigma\tilde{\alpha}^i(I_\varepsilon, II_\varepsilon, III_\varepsilon, \theta)$; $i = 0, 1, 2$ in Taylor series in I_ε, II_ε, III_ε, and θ about a known configuration $\underline{\Omega}$ and retain only up to linear terms in the invariants I_ε, II_ε, III_ε and θ and substitute these back in (2.53) or (2.57) and collect coefficients of the terms in the current configuration in terms of those in known configuration $\underline{\Omega}$. Let (for simplicity of notation)

$$ {}^\sigma\underset{\sim}{I}^1 = I_\varepsilon \; ; \quad {}^\sigma\underset{\sim}{I}^2 = II_\varepsilon \; ; \quad {}^\sigma\underset{\sim}{I}^3 = III_\varepsilon \tag{2.58} $$

Then the following is obtained for ${}_s\boldsymbol{\sigma}$

$$ \begin{aligned} {}_s\boldsymbol{\sigma} = {}& {}^0\bar{\sigma}\big|_{\underline{\Omega}} \boldsymbol{I} + {}^\sigma\underline{b}_1 \boldsymbol{\varepsilon} + {}^\sigma\underline{b}_2 \boldsymbol{\varepsilon}^2 + \sum_{j=1}^{3} {}^\sigma\underline{a}_j \left({}^\sigma\underset{\sim}{I}^j \boldsymbol{I} \right) + \sum_{j=1}^{3} {}^\sigma\underline{c}_{1j} \left({}^\sigma\underset{\sim}{I}^j \boldsymbol{\varepsilon} \right) \\ & + \sum_{j=1}^{3} {}^\sigma\underline{c}_{2j} \left({}^\sigma\underset{\sim}{I}^j \boldsymbol{\varepsilon}^2 \right) + {}^\sigma\underline{d}_1 \left((\theta - \theta_{\underline{\Omega}}) \boldsymbol{\varepsilon} \right) + {}^\sigma\underline{d}_2 \left((\theta - \theta_{\underline{\Omega}}) \boldsymbol{\varepsilon}^2 \right) \\ & - \underline{\alpha}_{\text{tm}} \left((\theta - \theta_{\underline{\Omega}}) \boldsymbol{I} \right) \end{aligned} \tag{2.59} $$

in which

$$
\begin{aligned}
{}^{0}\bar{\sigma}\big|_{\underline{\Omega}} &= {}^{\sigma}\underline{b}_0 & {}^{\sigma}\underline{a}_j &= \left.\frac{\partial({}^{\sigma}\tilde{\alpha}^0)}{\partial({}^{\sigma}\underset{\sim}{I}^j)}\right|_{\underline{\Omega}};\ j=1,2,3 \\
{}^{\sigma}\underline{b}_i = ({}^{\sigma}\tilde{\alpha}^i)\big|_{\underline{\Omega}} - \sum_{j=1}^{3} &\left.\frac{\partial({}^{\sigma}\tilde{\alpha}^i)}{\partial({}^{\sigma}\underset{\sim}{I}^j)}\right|_{\underline{\Omega}} ({}^{\sigma}\underset{\sim}{I}^j)_{\underline{\Omega}};\ i=0,1,2 & {}^{\sigma}\underline{c}_{1j} &= \left.\frac{\partial({}^{\sigma}\tilde{\alpha}^1)}{\partial({}^{\sigma}\underset{\sim}{I}^j)}\right|_{\underline{\Omega}};\ j=1,2,3 \\
{}^{\sigma}\underline{c}_{2j} &= \left.\frac{\partial({}^{\sigma}\tilde{\alpha}^2)}{\partial({}^{\sigma}\underset{\sim}{I}^j)}\right|_{\underline{\Omega}};\ j=1,2,3 & {}^{\sigma}\underline{d}_1 &= \left.\frac{\partial({}^{\sigma}\tilde{\alpha}^1)}{\partial\theta}\right|_{\underline{\Omega}} \\
{}^{\sigma}\underline{d}_2 &= \left.\frac{\partial({}^{\sigma}\tilde{\alpha}^2)}{\partial\theta}\right|_{\underline{\Omega}} & \underline{\alpha}_{\rm tm} &= -\left.\frac{\partial({}^{\sigma}\tilde{\alpha}^0)}{\partial\theta}\right|_{\underline{\Omega}}
\end{aligned}
\tag{2.60}
$$

The constitutive theory (2.59) for ${}_s\boldsymbol{\sigma}$ is based on integrity. Its simplified forms will be considered in later sections. The constitutive theory (2.59) requires 14 material coefficients.

2.4.5.5 Constitutive theory for ${}_s\boldsymbol{m}$

Following similar approach as used for ${}_s\sigma$ in section 2.4.5.3, either $\Phi = \Phi(\overset{\Theta}{\underset{s}{}}\boldsymbol{J}, \theta)$ or $\Phi = \Phi(I_{(\overset{\Theta}{\underset{s}{}}J)}, II_{(\overset{\Theta}{\underset{s}{}}J)}, III_{(\overset{\Theta}{\underset{s}{}}J)}, \theta)$ can be chosen and then (2.47) and Cayley-Hamilton Theorem can be used to obtain

$$
{}_s\boldsymbol{m} = {}^{m}\tilde{\alpha}^0\boldsymbol{I} + {}^{m}\tilde{\alpha}^1(\overset{\Theta}{\underset{s}{}}\boldsymbol{J}) + {}^{m}\tilde{\alpha}^2(\overset{\Theta}{\underset{s}{}}\boldsymbol{J})^2 \tag{2.61}
$$

in which

$$
{}^{m}\tilde{\alpha}^i = {}^{m}\tilde{\alpha}(I_{(\overset{\Theta}{\underset{s}{}}J)}, II_{(\overset{\Theta}{\underset{s}{}}J)}, III_{(\overset{\Theta}{\underset{s}{}}J)}, \theta); \quad i = 0,1,2 \tag{2.62}
$$

Alternatively one can use,

$$
{}_s\boldsymbol{m} = {}_s\boldsymbol{m}(\overset{\Theta}{\underset{s}{}}\boldsymbol{J}, \theta) \tag{2.63}
$$

and the representation theorem using $\boldsymbol{I}$, $\overset{\Theta}{\underset{s}{}}\boldsymbol{J}$,$(\overset{\Theta}{\underset{s}{}}\boldsymbol{J})^2$ as generators of $\overset{\Theta}{\underset{s}{}}\boldsymbol{J}$, θ that are symmetric tensors of rank two. A linear combination of these generators using ${}^{m}\tilde{\alpha}^i$; $i = 0,1,2$ defines the constitutive theory (2.61) for ${}_s\boldsymbol{m}$. Material coefficients in (2.61) are derived using (2.62), by considering Taylor series expansion of ${}^{m}\tilde{\alpha}^i$;$i = 0,1,2$ in $I_{(\overset{\Theta}{\underset{s}{}}J)}, II_{(\overset{\Theta}{\underset{s}{}}J)}, III_{(\overset{\Theta}{\underset{s}{}}J)}$ and θ about a known configuration $\underline{\Omega}$ and retaining only upto linear terms in the invariants and temperature θ. These are substituted in (2.61) and one collect coefficients (those derived in known configuration $\underline{\Omega}$) of the terms defined in current configuration. The final constitutive theory for ${}_s\boldsymbol{m}$ is given by

$$
\begin{aligned}
[m] = {}& {}^0\bar{m}\big|_{\underline{\Omega}}\,[I] + {}^m\underline{b}_1[{}^{\Theta}_{s}J] + {}^m\underline{b}_2[{}^{\Theta}_{s}J]^2 + \sum_{j=1}^{3} {}^m\underline{a}_j \left({}^m\underset{\sim}{I}^j[I]\right) + \sum_{j=1}^{3} {}^m\underline{c}_{1j} \left({}^m\underset{\sim}{I}^j[{}^{\Theta}_{s}J]\right) \\
& + \sum_{j=1}^{3} {}^m\underline{c}_{2j} \left({}^m\underset{\sim}{I}^j[{}^{\Theta}_{s}J]^2\right) + {}^m\underline{d}_1 \left((\theta-\theta_{\underline{\Omega}})[{}^{\Theta}_{s}J]\right) + {}^m\underline{d}_2 \left((\theta-\theta_{\underline{\Omega}})[{}^{\Theta}_{s}J]^2\right) \\
& - \underline{\alpha}_{\mathrm{tm}}\left((\theta-\theta_{\underline{\Omega}})[I]\right)
\end{aligned}
\tag{2.64}
$$

in which

$$
\begin{gathered}
{}^m\underset{\sim}{I}^1 = I_{({}^{\Theta}_{s}J)}, \quad {}^m\underset{\sim}{I}^2 = II_{({}^{\Theta}_{s}J)}, \text{ and } \ {}^m\underset{\sim}{I}^3 = III_{({}^{\Theta}_{s}J)} \\
{}^0\bar{m}\big|_{\underline{\Omega}} = \underline{b}_0 \qquad {}^m\underline{a}_j = \left.\frac{\partial({}^m\tilde{\alpha}^0)}{\partial({}^m\underset{\sim}{I}^j)}\right|_{\underline{\Omega}};\ j=1,2,3 \\
{}^m\underline{b}_i = ({}^m\tilde{\alpha}^i)\big|_{\underline{\Omega}} - \sum_{j=1}^{3} \left.\frac{\partial({}^m\tilde{\alpha}^i)}{\partial({}^m\underset{\sim}{I}^j)}\right|_{\underline{\Omega}} ({}^m\underset{\sim}{I}^j)_{\underline{\Omega}};\ i=0,1,2 \qquad {}^m\underline{c}_{1j} = \left.\frac{\partial({}^m\tilde{\alpha}^1)}{\partial({}^m\underset{\sim}{I}^j)}\right|_{\underline{\Omega}};\ j=1,2,3 \\
{}^m\underline{c}_{2j} = \left.\frac{\partial({}^m\tilde{\alpha}^2)}{\partial({}^m\underset{\sim}{I}^j)}\right|_{\underline{\Omega}};\ j=1,2,3 \qquad {}^m\underline{d}_1 = \left.\frac{\partial({}^m\tilde{\alpha}^1)}{\partial\theta}\right|_{\underline{\Omega}} \\
{}^m\underline{d}_2 = \left.\frac{\partial({}^m\tilde{\alpha}^2)}{\partial\theta}\right|_{\underline{\Omega}} \qquad \underline{\alpha}_{\mathrm{tm}} = -\left.\frac{\partial({}^m\tilde{\alpha}^0)}{\partial\theta}\right|_{\underline{\Omega}}
\end{gathered}
\tag{2.65}
$$

This constitutive theory requires determination of 14 material coefficients defined in (2.65), all evaluated in a known configuration $\underline{\Omega}$. Constitutive theory (2.64) is the most general and complete constitutive theory for $[m]$ as it is based on integrity. Simplified constitutive theories for $\boldsymbol{m}$ are presented in a subsequent section.

2.4.5.6 Constitutive theory for antisymmetric moment tensor ${}_a\boldsymbol{m}$

Constitutive theory for antisymmetric moment tensor ${}_a\boldsymbol{m}$ can be derived: (1) either using (2.48) and considering Φ as a function of the invariants of ${}^{\Theta}_{a}\boldsymbol{J}$ i.e, considering $\Phi = \Phi(I_{({}^{\Theta}_{a}J)}, II_{({}^{\Theta}_{a}J)}, III_{({}^{\Theta}_{a}J)}, \theta)$ or $\Phi = \Phi(i_{({}^{\Theta}_{a}J)}, \ddot{u}_{({}^{\Theta}_{a}J)}, \dddot{u}_{({}^{\Theta}_{a}J)}, \theta)$ (2) or by considering ${}_a\boldsymbol{m} = {}_a\boldsymbol{m}({}^{\Theta}_{a}\boldsymbol{J}, \theta)$ and using representation theorem. Both approach yield identically the same constitutive theory for ${}_a\boldsymbol{m}$. Noting,

$$
\begin{aligned}
I_{({}^{\Theta}_{a}J)} &= tr({}^{\Theta}_{a}J) = 0 \\
II_{({}^{\Theta}_{a}J)} &= \frac{1}{2}\left(\left(tr({}^{\Theta}_{a}J)\right)^2 - \left(tr({}^{\Theta}_{a}J)^2\right)\right) = \frac{1}{2}\left(tr({}^{\Theta}_{a}J)^2\right) \neq 0 \\
III_{({}^{\Theta}_{a}J)} &= det({}^{\Theta}_{a}J) = 0
\end{aligned}
\tag{2.66}
$$

and

$$
\begin{aligned}
i_{({}^{\Theta}_{a}J)} &= tr({}^{\Theta}_{a}J) = 0 \\
\ddot{u}_{({}^{\Theta}_{a}J)} &= \left(tr({}^{\Theta}_{a}J)^2\right) \neq 0 \\
\dddot{u}_{({}^{\Theta}_{a}J)} &= tr\left(({}^{\Theta}_{a}J)^3\right) = 0
\end{aligned}
\tag{2.67}
$$

Thus in this case the non zero invariants are $II_{({}^{\Theta}_{a}J)}$ and $\ddot{u}_{({}^{\Theta}_{a}J)}$ and furthermore

$$
II_{({}^{\Theta}_{a}J)} = -\frac{1}{2}\ddot{u}_{({}^{\Theta}_{a}J)}
\tag{2.68}
$$

Let ${}^{am}\underset{\sim}{I}^1$ be the nonzero invariant of (either one of the two choices) ${}_a\boldsymbol{m}$. Choice of $II_{({}^{\Theta}_{a}J)}$ or $\ddot{u}_{({}^{\Theta}_{a}J)}$ is irrelevant as there is a factor of $\frac{1}{2}$ between them. Let us choose

$$
{}^{am}\underset{\sim}{I}^1 = \left(tr({}^{\Theta}_{a}J)^2\right)
\tag{2.69}
$$

Now consider $\Phi = \Phi({}^{am}\underset{\sim}{I}^1, \theta)$ and use (2.48) to derive the following constitutive theory for ${}_a\boldsymbol{m}$

$$
{}_a\boldsymbol{m} = {}^{am}\alpha({}^{\Theta}_{a}J)
\tag{2.70}
$$

in which

$$
{}^{am}\alpha = {}^{am}\alpha({}^{am}\underset{\sim}{I}^1, \theta) = 2\rho_0 \frac{\partial\ \Phi({}^{am}\underset{\sim}{I}^1, \theta)}{\partial({}^{am}\underset{\sim}{I}^1)}
\tag{2.71}
$$

Alternatively one can consider

$$
{}_a\boldsymbol{m} = {}_a\boldsymbol{m}({}^{\Theta}_{a}\boldsymbol{J}, \theta)
\tag{2.72}
$$

Combined generators of ${}_{a}^{\Theta}\boldsymbol{J}, \theta$ that are antisymmetric tensors of rank two are only ${}_{a}^{\Theta}\boldsymbol{J}$. Hence we can express ${}_{a}\boldsymbol{m}$ as a linear combination of ${}_{a}^{\Theta}\boldsymbol{J}$ (using coefficient ${}^{am}\alpha$; same as in (2.70)) i.e,

$$
{}_{a}\boldsymbol{m} = {}^{am}\alpha({}_{a}^{\Theta}\boldsymbol{J}) \tag{2.73}
$$

which is same as (2.70). Material coefficients in (2.70) or (2.73) are derived by expanding ${}^{am}\alpha({}^{am}I^1, \theta)$ in Taylor series in ${}^{am}I^1$ and θ about a known configuration $\underline{\Omega}$ and retaining only upto linear terms in ${}^{am}I^1$ and θ.

$$
{}^{am}\alpha = {}^{am}\alpha\big|_{\underline{\Omega}} + \left.\frac{\partial({}^{am}\alpha)}{\partial({}^{am}\underset{\sim}{I}^1)}\right|_{\underline{\Omega}} ({}^{am}\underset{\sim}{I}^1 - {}^{am}\underset{\sim}{I}^1\big|_{\underline{\Omega}}) - \left.\frac{\partial({}^{am}\alpha)}{\partial\theta}\right|_{\underline{\Omega}} (\theta - \theta|_{\underline{\Omega}}) \tag{2.74}
$$

Substituting (2.74) in (2.70) or (2.73) and collecting coefficients, one obtains

$$
{}_{a}\boldsymbol{m} = {}^{am}\underline{b}_1({}_{a}^{\Theta}J) + {}^{am}\underline{c}_{11}({}^{am}\underset{\sim}{I}^1)({}_{a}^{\Theta}J) - {}^{am}\underline{d}_2(\theta - \theta|_{\underline{\Omega}})({}_{a}^{\Theta}J) \tag{2.75}
$$

in which

$$
\begin{aligned}
{}^{am}\underline{b}_1 &= {}^{am}\alpha\big|_{\underline{\Omega}} - \frac{\partial({}^{am}\alpha)}{\partial({}^{am}\underset{\sim}{I}^1)} ({}^{am}\underset{\sim}{I}^1)\big|_{\underline{\Omega}} \\
{}^{am}\underline{c}_{11} &= \left.\frac{\partial({}^{am}\alpha)}{\partial({}^{am}\underset{\sim}{I}^1)}\right|_{\underline{\Omega}} \\
{}^{am}\underline{d}_2 &= \left.\frac{\partial({}^{am}\alpha)}{\partial\theta}\right|_{\underline{\Omega}}
\end{aligned} \tag{2.76}
$$

This constitutive theory (2.75) for ${}_{a}\boldsymbol{m}$ requires only three material coefficients. If $(\theta - \theta|_{\underline{\Omega}})$ terms are neglected, then one only needs two material coefficients ${}^{am}\underline{b}_1$ and ${}^{am}\underline{c}_{11}$. This constitutive theory contains upto cubic terms in the components of ${}_{a}^{\Theta}\boldsymbol{J}$. This constitutive theory is based on integrity.

2.5 Conservation and balance laws and the constitutive theories when balance of moments of moments is a balance law

Necessity of the balance of moments of moments balance law from theoretical stand point and the consequences of its absence in the mathematical model will be considered in section 2.6. In this section one presents the outcome of considering this balance law in terms of influencing the mathematical model as well as constitutive theories. The conservation and the balance laws that are not affected due to consideration of balance of moments of moments balance law are not repeated in the following for the sake of brevity. Conservation of mass (2.16), balance of linear momenta (2.18), balance of angular momenta (2.26), energy equation (2.35) and the form of the entropy inequality (2.38) hold regardless of whether balance of moments of moments is a balance law or not. In the following one first presents derivation of the balance of moments of moments balance law.

2.5.1 Balance of moments of moments balance law

Consider current configuration at time t. For the deforming volume of matter to be in equilibrium, the moments of moments (or couples) must vanish. In the moment of moments one must consider $\bar{\boldsymbol{M}}$ and also the shear components of the stress tensor $\bar{\boldsymbol{\sigma}}$, that is, $\boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}}$. Thus, one can write (neglecting inertial terms) in Eulerian description:

$$\int\limits_{\bar{V}} \bar{\boldsymbol{x}} \times (\boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}})\, d\bar{V} - \int\limits_{\partial\bar{V}} \bar{\boldsymbol{x}} \times \bar{\boldsymbol{M}}\, d\bar{A} = 0 \tag{2.77}$$

Expanding the second term in (2.77) and then converting the integral over $\partial\bar{V}$ to the integral over $\bar{V}$ using the divergence theorem:

$$
\begin{aligned}
\int_{\partial \bar{V}} \bar{\boldsymbol{x}} \times \bar{\boldsymbol{M}}\, d\bar{A} &= \int_{\partial \bar{V}} \boldsymbol{e}_k \epsilon_{ijk} x_i \bar{M}_j \, d\bar{A} \\
&= \int_{\partial \bar{V}} \boldsymbol{e}_k \epsilon_{ijk} \bar{x}_i \bar{m}_{mj} \bar{n}_m \, d\bar{A} \\
&= \int_{\bar{V}} \boldsymbol{e}_k (\epsilon_{ijk} \bar{x}_i \bar{m}_{mj})_{,m} \, d\bar{V} \\
&= \int_{\bar{V}} \boldsymbol{e}_k \epsilon_{ijk} (\bar{m}_{ij} + \bar{x}_i \bar{m}_{mj,m}) \, d\bar{V} \\
&= \int_{\bar{V}} \boldsymbol{e}_k \epsilon_{ijk} \bar{m}_{ij} \, d\bar{V} + \int_{\bar{V}} \bar{\boldsymbol{x}} \times \left(\bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{m}} \right) \, d\bar{V}
\end{aligned}
\tag{2.78}
$$

Using equation (2.78) in (2.77) and collecting terms

$$
\int_{\bar{V}} \bar{\boldsymbol{x}} \times \left(-\bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{m}} + \boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}} \right) d\bar{V} - \int_{\bar{V}} \boldsymbol{e}_k \epsilon_{ijk} \bar{m}_{ij} \, d\bar{V} = 0 \tag{2.79}
$$

The first term in (2.79) vanishes due to balance of angular momenta, giving the condition

$$
\int_{\bar{V}} \boldsymbol{e}_k \epsilon_{ijk} \bar{m}_{ij} \, d\bar{V} = 0 \tag{2.80}
$$

which, because $\bar{V}$ is arbitrary, yields

$$
\epsilon_{ijk} \bar{m}_{ij} = 0 \qquad \text{and} \qquad \epsilon_{ijk} m_{ij} = 0 \tag{2.81}
$$

Equation (2.81) implies that Cauchy moment tensor $\boldsymbol{m}$ is symmetric in non-classical continuum theories when balance of moments of moments is used as an additional balance law.

Remarks.

(1) One can note that $\boldsymbol{m}$ in balance of angular momenta (2.26) was nonsymmetric (section 2.4),

but now it is symmetric due to consideration of balance of moments of moments as a balance law.

(2) In the energy equation (2.35) as well as entropy inequality (2.38), $\boldsymbol{m}$ is symmetric as well. Thus one needs to reconsider (2.35) and (2.38) in view of the fact that $\boldsymbol{m}$ is symmetric.

(3) One may also need some new considerations in the derivation of the constitutive theories. These are considered in the following.

2.5.2 Entropy inequality and the energy equation

Consider entropy inequality (2.38) derived in section 2.4 (in the absence of balance of moments of moments as a balance law)

$$\rho_0 \left(\frac{D\Phi}{Dt} + \eta \frac{D\theta}{Dt} \right) + \frac{\boldsymbol{q} \cdot \boldsymbol{g}}{\theta} - \text{tr}\left([\sigma][{}^{d}\dot{J}] \right) - \text{tr}\left([m][{}^{\Theta}\dot{J}] \right) - {}_{i}\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{m}) \leq 0 \tag{2.82}$$

in which $\boldsymbol{m}$ is now a symmetric cauchy moment tensor. One considers the following decompositions (as in equation (2.40))

$$\begin{aligned} \boldsymbol{\sigma} &= {}_{s}\boldsymbol{\sigma} + {}_{a}\boldsymbol{\sigma} \\ \boldsymbol{m} &= {}_{s}\boldsymbol{m} \; ; \quad {}_{a}\boldsymbol{m} = 0 \\ {}^{d}\dot{\boldsymbol{J}} &= {}^{d}_{s}\dot{\boldsymbol{J}} + {}^{d}_{a}\dot{\boldsymbol{J}} \\ {}^{\Theta}\dot{\boldsymbol{J}} &= {}^{\Theta}_{s}\dot{\boldsymbol{J}} + {}^{\Theta}_{a}\dot{\boldsymbol{J}} \end{aligned} \tag{2.83}$$

Substituting (2.83) in (2.82) and noting that

$$\text{tr}\left([{}_{s}\sigma][{}^{d}_{a}\dot{J}] \right) = 0; \quad \text{tr}\left([{}_{a}\sigma][{}^{d}_{s}\dot{J}] \right) = 0; \quad \text{tr}\left([m][{}^{\Theta}_{a}\dot{J}] \right) = 0 \tag{2.84}$$

and

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \boldsymbol{m} &= \boldsymbol{\epsilon} : \boldsymbol{\sigma} \\
{}_{i}\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{m}) &= {}_{i}\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\epsilon} : \boldsymbol{\sigma}) \\
\text{tr}\left([{}_{a}\sigma][{}^{d}_{a}\dot{J}]\right) &= -{}_{i}\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\epsilon} : \boldsymbol{\sigma})
\end{aligned}
\tag{2.85}
$$

One obtains,

$$
\rho_0 \left(\frac{D\Phi}{Dt} + \eta \frac{D\theta}{Dt} \right) + \frac{\boldsymbol{q} \cdot \boldsymbol{g}}{\theta} - \text{tr}\left([{}_{s}\sigma][{}^{d}_{s}\dot{J}]\right) - \text{tr}\left([m][{}^{\Theta}_{s}\dot{J}]\right) \leq 0 \tag{2.86}
$$

In (2.86), $\boldsymbol{m}$ is same as ${}_{s}\boldsymbol{m}$ since ${}_{a}\boldsymbol{m} = 0$. Comparing (2.86) with (2.43) in section 2.4, one notes that when the balance of moments of moments is not used as a balance law $({}_{a}\boldsymbol{m}, {}^{\Theta}_{a}\dot{\boldsymbol{J}})$ appears as additional conjugate pair in rate of work necessitating constitutive theory for ${}_{a}m$. This constitutive theory obviously does not exist when balance of moments of moments is used as a balance law. The energy equation (2.44) modifies accordingly and one gets

$$
\rho_0 \frac{De}{Dt} + \boldsymbol{\nabla} \cdot \boldsymbol{q} - \text{tr}([{}_{s}\sigma][{}^{d}_{s}\dot{J}]) - \text{tr}([{}_{s}m][{}^{\Theta}_{s}\dot{J}]) = 0 \tag{2.87}
$$

2.5.3 Constitutive theories

Following section 2.4, one can write

$$
\begin{aligned}
\Phi &= \Phi({}^{d}_{s}\boldsymbol{J}, {}^{\Theta}_{s}\boldsymbol{J}, \boldsymbol{g}, \theta) \\
\eta &= \eta({}^{d}_{s}\boldsymbol{J}, {}^{\Theta}_{s}\boldsymbol{J}, \boldsymbol{g}, \theta) \\
{}_{s}\boldsymbol{\sigma} &= {}_{s}\boldsymbol{\sigma}({}^{d}_{s}\boldsymbol{J}, \theta) \\
\boldsymbol{m} &= \boldsymbol{m}({}^{\Theta}_{s}\boldsymbol{J}, \theta) = {}_{s}\boldsymbol{m} \\
\boldsymbol{q} &= \boldsymbol{q}(\boldsymbol{g}, \theta)
\end{aligned}
\tag{2.88}
$$

Using Φ and its arguments in (2.88), $\dot{\Phi}$ is determined and this is substituted in the entropy inequality (2.44) (following same procedure as in section 2.4) to arrive at

$$ {}_s\boldsymbol{\sigma} = \rho_0 \frac{\partial \phi}{\partial({}^d_s\boldsymbol{J})} \tag{2.89} $$

$$ {}_s m_{ki} = \rho_0 \frac{\partial \phi}{\partial({}^\Theta_s J_{ki})} \tag{2.90} $$

$$ {}_a\boldsymbol{m} = \rho_0 \frac{\partial \phi}{\partial({}^\Theta_a\boldsymbol{J})} \tag{2.91} $$

$$ \eta = -\frac{\partial \Phi}{\partial \theta} \tag{2.92} $$

$$ \Phi \neq \Phi(\boldsymbol{g}) \tag{2.93} $$

$$ \text{and} \qquad \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} \leq 0 \tag{2.94} $$

Derivations of the constitutive theories for ${}_s\boldsymbol{\sigma}$ and $\boldsymbol{m}$ follow exactly same procedure as in section 2.4 for ${}_s\boldsymbol{\sigma}$ and ${}_s\boldsymbol{m}$ (not repeated here). One notes that when balance of moments of moments is considered as a balance law: (1) $\boldsymbol{m}$ is symmetric i.e $\boldsymbol{m} = {}_s\boldsymbol{m}$ and ${}_a\boldsymbol{m} = 0$. (2) hence, there is no constitutive theory for ${}_a\boldsymbol{m}$.

2.6 Necessity of the "balance of moments of moments" balance law in non-classical continuum theories

One first considers inductive reasoning as to why this balance law is needed in non-classical continuum theories. In case of classical continuum theories that consider only forces and displacements at material points, it is well known that equilibrium considerations between actions and reactions for stable equilibrium of the deformed volume of matter require (i) the sum of all forces to be zero in a fixed frame (balance of linear momenta) and (ii) the sum of moments of all forces constituting actions and reactions about any point should vanish (balance of angular momenta).

These two balance laws ensure than an elementary tetrahedron with actions (forces) on its oblique plane and reactions (Cauchy stress tensor) on its orthogonal planes will be in stable equilibrium. The consequence of the balance of angular momenta is that the Cauchy stress tensor is symmetric. Thus, one notes that when forces and displacements are the only considerations at a material point as in classical continuum theories, balance of forces and balance of their moments are essential for equilibrium considerations.

Using the equilibrium consideration for a single material particle encased in a volume element as a basis, Yang et al. [6] established the equilibrium of rotations for a system of material particles to account for the equilibrium of the moments of couples (or moments) of a body. *When a system of forces are applied to a system of multiple particles the equilibrium relations are derived from a resultant force and a resultant couple of forces applied to an arbitrary point. The couple of moment of forces is a free vector in classical mechanics, which means that the effect of the couple applied on an arbitrary point in the space of the system of materials particles is independent of the position of the point. In other words, the couple can translate to any point in the space freely and the resulting effects are unchanged. As a result, only the conventional force equilibrium and the moment equilibrium (balance of linear and angular momenta) are involved in the equilibrium relations [22–24]. Equivalence of a couple resulting from the rotations ${}_i\boldsymbol{\Theta}$ that is not a free vector but a driving force that rotates the material particles requires considerations (see [6] for derivation) that eventually results into balance of moments of moments or couples for static equilibrium.* This balance law is an extension of balance of angular momenta in classical mechanics to non-classical mechanics in which moments due to rotations ${}_i\boldsymbol{\Theta}$ exist in addition to displacements.

Balance of moments of moments (similar to balance of moment of forces in classical continuum theories) is additional balance law needed due to presence of Cauchy moment tensor that is independent of forces. In the derivation presented in section 2.5.1, one notes that this balance law yields symmetry of the Cauchy moment tensor, just like balance of angular momentum in classical continuum mechanics gives rise to symmetry of the Cauchy moment tensor. One can use inductive reasoning to extend this concept of the need for additional balance laws when additional kinematic variables (over and beyond displacements and rotations) and their conjugates appear in the theory. One notes that each additional kinematic variable introduces its conjugate that requires two balance laws out of which the law that requires their sum to balance with others already exists from consideration of prior kinematic variables; hence, the new conjugate quantities can be incorporated in it, but the other law that requires balance of their moments is an additional balance law. In other words, only one balance law is needed for each conjugate quantity corresponding to each new kinematic variable. In the present non-classical continuum theory one additional balance law is needed, namely, the balance of moment of moments, due to the fact that the balance of angular momenta already exist from the classical continuum theory.

In the following one considers linear constitutive theories (due to simplicity) and also due to ease of highlighting the differences when balance of moments of moment is used or not used as a balance law. Constitutive theories for $\boldsymbol{q}$ are not included here. These do not influence our discussion regarding the balance of moments of moments balance law.

2.6.1 Complete mathematical model including linear constitutive theories when balance of moments of moments is not a balance law

In this case Cauchy moment tensor $\boldsymbol{m}$ is not symmetric.

$$\rho_0 = |J|\rho(\boldsymbol{x},t) \tag{2.95}$$

$$\rho_0 \frac{D\boldsymbol{v}}{Dt} - \rho_0 \boldsymbol{F}^b - \boldsymbol{\nabla}\cdot{}_s\boldsymbol{\sigma} - \boldsymbol{\nabla}\cdot{}_a\boldsymbol{\sigma} = 0 \tag{2.96}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{m} - \boldsymbol{\epsilon} : \boldsymbol{\sigma} = 0 \tag{2.97}$$

$$\rho_0 \frac{De}{Dt} + \boldsymbol{\nabla}\cdot\boldsymbol{q} - \text{tr}\left([{}_s\sigma][{}^d_s\dot{J}]\right) - \text{tr}\left([{}_sm][{}^\Theta_s\dot{J}]\right) - -\text{tr}\left([{}_am][{}^\Theta_a\dot{J}]\right) = 0 \tag{2.98}$$

$$\rho_0\left(\frac{D\Phi}{Dt} + \eta\frac{D\theta}{Dt}\right) + \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} - \text{tr}\left([{}_s\sigma][{}^d_s\dot{J}]\right) - \text{tr}\left([{}_sm][{}^\Theta_s\dot{J}]\right) - \text{tr}\left([{}_am][{}^\Theta_a\dot{J}]\right) \leq 0 \tag{2.99}$$

$$
{}_s\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda \mathrm{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} \; ; \qquad ({}^{\sigma}\underline{b}_1 = 2\mu, {}^{\sigma}\underline{a}_1 = \lambda, {}^{d}_{s}\boldsymbol{J} = \boldsymbol{\varepsilon}) \tag{2.100}
$$

$$
{}_s\boldsymbol{m} = \alpha({}^{\Theta}_{s}\boldsymbol{J}) \; ; \qquad ({}^{m}\underline{b}_1 = \alpha \text{ and } \mathrm{tr}({}^{\Theta}_{s}\boldsymbol{J}) = 0) \tag{2.101}
$$

$$
{}_a\boldsymbol{m} = \beta({}^{\Theta}_{a}\boldsymbol{J}); \quad {}^{am}b_1 = \beta \tag{2.102}
$$

This mathematical model also has closure and requires μ, λ ,α and β as four material coefficients.

2.6.2 Complete mathematical model including linear constitutive theory when balance of moments of moments is a balance law

In this case Cauchy moment tensor $\boldsymbol{m}$ is symmetric.

$$
\rho_0 = |J|\rho(\boldsymbol{x}, t) \tag{2.103}
$$

$$
\rho_0 \frac{D\boldsymbol{v}}{Dt} - \rho_0 \boldsymbol{F}^b - \boldsymbol{\nabla} \cdot {}_s\boldsymbol{\sigma} - \boldsymbol{\nabla} \cdot {}_a\boldsymbol{\sigma} = 0 \tag{2.104}
$$

$$
\boldsymbol{\nabla} \cdot \boldsymbol{m} - \boldsymbol{\epsilon} : \boldsymbol{\sigma} = 0 \tag{2.105}
$$

$$
\epsilon_{ijk} m_{ij} = 0; \qquad \text{due to balance of moments of moments balance law} \tag{2.106}
$$

$$
\rho_0 \frac{De}{Dt} + \boldsymbol{\nabla} \cdot \boldsymbol{q} - \mathrm{tr}\left([{}_s\sigma][{}^{d}_{s}\dot{J}]\right) - \mathrm{tr}\left([m][{}^{\Theta}_{s}\dot{J}]\right) = 0 \tag{2.107}
$$

$$
\rho_0 \left(\frac{D\Phi}{Dt} + \eta\frac{D\theta}{Dt}\right) + \frac{\boldsymbol{q} \cdot \boldsymbol{g}}{\theta} - \mathrm{tr}\left([{}_s\sigma][{}^{d}_{s}\dot{J}]\right) - \mathrm{tr}\left([m][{}^{\Theta}_{s}\dot{J}]\right) \leq 0 \tag{2.108}
$$

$$
{}_s\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda \mathrm{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} \tag{2.109}
$$

$$
\boldsymbol{m} = \alpha({}^{\Theta}_{s}\boldsymbol{J}) \tag{2.110}
$$

This mathematical model also has closure and requires only three material coefficients μ, λ and α. Absence of material coefficient β (compared to section 2.6.1) is of course due to the fact that in this case $\boldsymbol{m}$ is symmetric hence ${}_a\boldsymbol{m} = 0$, thus no constitutive theory for ${}_a\boldsymbol{m}$ that contains material coefficient β.

2.7 Lack of validity of the constitutive theories for the Cauchy moment tensor when 'balance of moments of moments is not a balance law'

In this section one presents theoretical consideration as well as demonstration through simple model problems with simplified physics, regarding the lack of validity of the constitutive theories for the moment tensor in non-classical continuum mechanics when the balance of moments of moments is not considered a balance law.

2.7.1 Theoretical considerations

In this section one considers mathematical models and the linear constitutive theories in section 2.6.1 and 2.6.2. Consider the mathematical model in 2.6.1, in the absence of the balance of moments of moments as a balance law. One notes that

(1) Entropy inequality yields conjugate pairs $({}_s\boldsymbol{m}, \underset{s}{\overset{\Theta}{\dot{\boldsymbol{J}}}})$ and $({}_a\boldsymbol{m}, \underset{a}{\overset{\Theta}{\dot{\boldsymbol{J}}}})$, hence in this case it is essential to have constitutive theories for both ${}_s\boldsymbol{m}$ and ${}_a\boldsymbol{m}$.

(2) In some published works only the constitutive theory for ${}_a\boldsymbol{m}$ is advocated in connection with non-classical theory applications in solid mechanics (bending of plates). From conjugate pairs in entropy inequality, it is true that constitutive theory for ${}_a\boldsymbol{m}$ is required, however constitutive theory for ${}_s\boldsymbol{m}$ can not be discarded as this would result in violation of entropy inequality, hence violation of thermodynamic equilibrium during the deformation.

(3) From constitutive theories for ${}_s\boldsymbol{m}$ and ${}_a\boldsymbol{m}$ in (2.100) and (2.101)

$$\begin{aligned} {}_s\boldsymbol{m} &= \alpha({}^{\Theta}_{s}\boldsymbol{J}) \\ {}_a\boldsymbol{m} &= \beta({}^{\Theta}_{a}\boldsymbol{J}) \end{aligned} \tag{2.111}$$

Since ${}^{\Theta}_{s}\boldsymbol{J}$ and ${}^{\Theta}_{a}\boldsymbol{J}$ are both from the rotation gradient tensor, it is perhaps reasonable and may be instructive in the following if one assumes that $\alpha = \beta = \gamma$, then (2.111) can be written as

$$\begin{aligned} {}_s\boldsymbol{m} &= \gamma({}^{\Theta}_{s}\boldsymbol{J}) \\ {}_a\boldsymbol{m} &= \gamma({}^{\Theta}_{a}\boldsymbol{J}) \end{aligned} \tag{2.112}$$

which implies that

$$\boldsymbol{m} = ({}_s\boldsymbol{m} +_a \boldsymbol{m}) = \gamma({}^{\Theta}_{s}\boldsymbol{J} + {}^{\Theta}_{a}\boldsymbol{J}) = \gamma({}^{\Theta}\boldsymbol{J}) \tag{2.113}$$

One notes that Cauchy moment tensor $\boldsymbol{m}$ and the rotation gradient tensor ${}^{\Theta}\boldsymbol{J}$ both are non-symmetric tensors of rank two. Constitutive theory (2.113) implies that

$$\boldsymbol{m} = \boldsymbol{m}({}^{\Theta}\boldsymbol{J}, \theta) \tag{2.114}$$

Based on the representation theorem and the works of Spencer, Wang, Zheng, etc. [52–68], (2.114) is not possible. If $\boldsymbol{m}$, a non-symmetric tensor that exists in a space, then its basis must be deterministic from its argument tensors ${}^{\Theta}\boldsymbol{J}$ and θ in which ${}^{\Theta}\boldsymbol{J}$ is also non-symmetric tensor of rank two and θ is a tensor of rank zero. But this is not possible when the constitutive tensor and its arguments are non-symmetric tensors.

(4) From (3) one concludes that (2.113) and (2.114) are not valid. One can re-examine (2.111) and (2.112). From the works of Surana et al. [2–5, 7, 8] on internal polar non-classical theories, one notes that incorporating internal rotations and their rates in continuum theories results in additional resistance to deformations, hence resulting in reduced displacements, increased stress and moment tensor fields, increased energy storage and dissipation (in case

of thermoviscoelastic solids). The physics of internal rotations and conjugate moment tensor are mathematically consistent with the observed behavior in model problems as well but only in the absence of the constitutive theory for ${}_a\boldsymbol{m}$ which necessitates that one considers balance of moments of moments as a balance law in the non-classical theories incorporating internal rotations.

(5) The assumption of $\alpha = \beta = \gamma$ is essential in arriving at (2.114). One can also consider an alternate approach. One can assume both constitutive theories for ${}_s\boldsymbol{m}$ and ${}_a\boldsymbol{m}$ (in (2.111)) hold and arrive at contradiction. Consider $\alpha > 0$ and $\beta = 0$. In this case for progressively increasing value of α (but $\beta = 0$), the internal polar physics influence also becomes more dominant resulting in progressively reduced displacements, increase stress and moment field, increased energy storage etc. The influence of the constitutive theory for ${}_a\boldsymbol{m}$ is counter or opposite to ${}_s\boldsymbol{m}$ due to negative sign (also shown in model problems studies) i.e for a fixed value of α if β is progressively increased, then the influence of polar physics is progressively reduced and at $\beta = \alpha$, one has the polar physics completely eliminated.

(6) One considers a plane stress (or strain) model problem with internal polar physics to illustrate the serious adverse consequences when balance of moments of moments is not used as a balance law in non-classical continuum theories.

For simplicity of notations, if one chooses $x = x_1$, $y = x_2$, $u = u_x$, $v = u_{x_z}$ and likewise $\sigma_{ij} : \sigma_{xx}, \sigma_{xy}, \sigma_{yx}, \sigma_{yy}$ and $m_{ij} = m_{xx}, m_{xy}, m_{yx}, m_{yy}$, then using $\sigma_{ij} = {}_s\sigma_{ij} + {}_a\sigma_{ij}$ and $m_{ij} = {}_sm_{ij} + {}_am_{ij}$; $i, j = x, y$, one can write the following (in the absence of body forces and inertial terms) for the conservation and balance laws and the linear constitutive theories.

$$\frac{\partial_s\sigma_{xx}}{\partial x} + \frac{\partial_s\sigma_{yx}}{\partial y} + \frac{\partial_a\sigma_{yx}}{\partial y} = 0 \tag{2.115}$$

$$\frac{\partial_s\sigma_{xy}}{\partial x} + \frac{\partial_s\sigma_{yy}}{\partial y} - \frac{\partial_a\sigma_{yx}}{\partial x} = 0 \tag{2.116}$$

$$\frac{\partial m_{xz}}{\partial x} + \frac{\partial m_{yz}}{\partial y} + 2({}_a\sigma_{yx}) = 0 \tag{2.117}$$

$$
\left.\begin{aligned}
{}_s\sigma_{xx} = D_{11}\frac{\partial u}{\partial x} + D_{12}\frac{\partial u}{\partial y}; \quad {}_s\sigma_{yy} = D_{21}\frac{\partial u}{\partial x} + D_{22}\frac{\partial v}{\partial y} \\
\text{and} \qquad {}_s\sigma_{xy} = D_{33}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)
\end{aligned}\right\} \tag{2.118}
$$

Coefficients D_{ij} contain modulus of elasticity E and Poisson's ratio ν

$$
\begin{gathered}
\left[{}^{\Theta}J\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial({}_i\Theta_z)}{\partial x} & \frac{\partial({}_i\Theta_z)}{\partial y} & 0 \end{bmatrix}; \quad \left[{}^{\Theta}_{s}J\right] = \begin{bmatrix} 0 & 0 & \frac{1}{2}\frac{\partial({}_i\Theta_z)}{\partial x} \\ 0 & 0 & \frac{1}{2}\frac{\partial({}_i\Theta_z)}{\partial y} \\ \frac{1}{2}\frac{\partial({}_i\Theta_z)}{\partial x} & \frac{1}{2}\frac{\partial({}_i\Theta_z)}{\partial y} & 0 \end{bmatrix} \\
\left[{}^{\Theta}_{a}J\right] = \begin{bmatrix} 0 & 0 & -\frac{1}{2}\frac{\partial({}_i\Theta_z)}{\partial x} \\ 0 & 0 & -\frac{1}{2}\frac{\partial({}_i\Theta_z)}{\partial y} \\ \frac{1}{2}\frac{\partial({}_i\Theta_z)}{\partial x} & \frac{1}{2}\frac{\partial({}_i\Theta_z)}{\partial y} & 0 \end{bmatrix}
\end{gathered} \tag{2.119}
$$

a) Constitutive theory for ${}_s\boldsymbol{m}$ and ${}_a\boldsymbol{m}$

$$
\left[{}_s m\right] = \alpha \left[{}^{\Theta}_{s}J\right] \tag{2.120}
$$

Hence the constitutive theory for the only non-zero components are

$$
\begin{aligned}
{}_s m_{xz} = {}_s m_{zx} = \frac{\alpha}{2}\frac{\partial({}_i\Theta_z)}{\partial x} = \underset{\sim}{\alpha}\frac{\partial({}_i\Theta_z)}{\partial x} \\
{}_s m_{yz} = {}_s m_{zy} = \frac{\alpha}{2}\frac{\partial({}_i\Theta_z)}{\partial y} = \underset{\sim}{\beta}\frac{\partial({}_i\Theta_z)}{\partial y}
\end{aligned} \tag{2.121}
$$

$$
\left[{}_a m\right] = \beta \left[{}^{\Theta}_{a}J\right] \tag{2.122}
$$

Hence, the constitutive theories for the only nonzero components are

$$
{}_a m_{xz} = -\frac{\beta}{2}\frac{\partial({}_i\Theta_z)}{\partial x} = -\underset{\sim}{\beta}\frac{\partial({}_i\Theta_z)}{\partial x}\,; \qquad {}_a m_{yz} = -\frac{\beta}{2}\frac{\partial({}_i\Theta_z)}{\partial y} == -\underset{\sim}{\beta}\frac{\partial({}_i\Theta_z)}{\partial y} \tag{2.123}
$$

$$
\text{and} \quad {}_i\Theta_z = \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) \tag{2.124}
$$

The coefficient $D_{ij};\ i,j = 1,2$ can be easily obtained in terms of modulus of elasticity E and poisson's ratio ν or the Lame's constants μ and λ.

One also notes that from (2.122) and (2.123) one has

$$
{}_{a}m_{zx} = \underset{\sim}{\beta}\frac{\partial({}_{i}\Theta_z)}{\partial x} = -{}_{a}m_{xz}\ ; \qquad {}_{a}m_{zy} = \underset{\sim}{\beta}\frac{\partial({}_{i}\Theta_z)}{\partial y} = -{}_{a}m_{yz} \tag{2.125}
$$

Choice of $({}_{a}m_{xz}, {}_{a}m_{yz})$ or $({}_{a}m_{zx}, {}_{a}m_{zy})$ does not matter as there is only a sign change between these pairs and corresponding constitutive theories. In the following one considers (2.121) and (2.123) from which one can write

$$
\begin{aligned}
m_{xz} &= {}_{s}m_{xz} + {}_{a}m_{xz} = (\underset{\sim}{\alpha} - \underset{\sim}{\beta})\frac{\partial({}_{i}\Theta_z)}{\partial x} \\
m_{yz} &= {}_{s}m_{yz} + {}_{a}m_{yz} = (\underset{\sim}{\alpha} - \underset{\sim}{\beta})\frac{\partial({}_{i}\Theta_z)}{\partial y}
\end{aligned} \tag{2.126}
$$

(i) When $\underset{\sim}{\beta} = 0$, $m_{xz} = {}_{s}m_{xz}$, $m_{yz} = {}_{s}m_{yz}$, but this is only possible if ${}_{a}m_{yz} = 0$, implying that balance of moments of moments is a balance law.

(ii) From (2.126) one can see that for $\underset{\sim}{\beta} > 0$, as $\underset{\sim}{\beta}$ increases for a fixed $\underset{\sim}{\alpha}$, m_{xz} and m_{yz} reduce implying the progressively decreasing polar physics.

(iii) When $\underset{\sim}{\beta} = \underset{\sim}{\alpha}$, $m_{xz} = m_{yz} = 0$ everywhere in the spatial domain implying that their gradients (2.117) are zero which of course leads us to conclude that ${}_{a}\sigma = 0$, that is complete absence of internal polar physics. One notes that for $\underset{\sim}{\beta} = 0$, with progressively increasing values of $\underset{\sim}{\alpha}$ simulate progressively increasing polar physics i.e progressively stiffer behavior of the deforming matter. Progressively increasing values of $\underset{\sim}{\beta}$ imply weakening internal polar physics i.e softening of the deforming matter with total absence of polar physics when $\underset{\sim}{\beta} = \underset{\sim}{\alpha}$.

(iv) From (2.126) one notes that when one only has constitutive theory for ${}_{a}\boldsymbol{m}$ (as advocated in some published works) i.e $\underset{\sim}{\alpha} = 0$, the material will exhibit progressively softening behavior with progressively increasing values of $\underset{\sim}{\beta}$. This is precisely opposite of what the influence of internal rotation physics should be.

(v) In this section it is conclusively established through conservation and the balance laws that balance of moments of moments is an essential balance law in non-classical continuum theories incorporating internal rotation physics.

2.8 Model Problems

In this section one considers simple model problems in which classical continuum behavior is well understood so that the influence of internal polar physics associated with Cauchy moment tensors ${}_s\boldsymbol{m}$ and ${}_a\boldsymbol{m}$ on the deformation behavior can be clearly demonstrated. One considers conservation and balance laws in $\mathbb{R}^2$ for plane stress behavior. This mathematical model is used to study classical and internal polar physics and its influence on deformation of slender clamped-clamped plate and a simply supported plate. The mathematical models in the absence of the balance of moments of moments as a balance law and also in its presence have been described in section 2.6.1 and 2.6.2 An analytical solution of the equations in this model without significant simplifications is not possible, hence one considers numerical solution using p-version hierarchical finite elements with higher order global differentiability local approximations in $H^{k,p}(\bar{\Omega}^e)$ scalar product spaces in which the integral forms are constructed using residual functional (least square method) [84]. For this purpose the mathematical model needs to be non-dimensionalized.

2.8.1 Dimensionless form of the mathematical model in $\mathbb{R}^2$ for plane stress

Non-dimensionalizing the mathematical model presented in $\mathbb{R}^2$ for the plane stress case (equations (2.115)-(2.125)) which is the expanded form of the model in 2.6.1. One rewrite (2.115)-(2.125) with a hat ($\hat{\ }$) on all quantities indicating that the quantities have their usual dimensions in terms of length ($\hat{L}$), force ($\hat{F}$) and time ($\hat{t}$). If one chooses L_0, F_0 and t_0 as reference values of length, force, and time then the dimensionless length, force, and time (L, F and t) are defined as

$$L = \frac{\hat{L}}{L_0}, \quad F = \frac{\hat{F}}{F_0}, \quad t = \frac{\hat{t}}{t_0} \tag{2.127}$$

If one considers $\hat{E} = EE_0$, $\hat{x} = xL_0$, $\hat{y} = yL_0$, $\hat{m} = mm_0$, $m_0 = \frac{\tau_0}{L_0}$, $F_0 = \tau_0 L_0^2$, $\hat{\underset{\sim}{\alpha}} = \underset{\sim}{\alpha} E_0$, $\hat{\underset{\sim}{\beta}} = \underset{\sim}{\beta} E_0$ and choose L_0, E_0, then the dimensionless form of the mathematical model can be written as

$$\begin{gathered}\frac{\partial_s \sigma_{xx}}{\partial x} + \frac{\partial_s \sigma_{yx}}{\partial y} + \frac{\partial_a \sigma_{yx}}{\partial y} = 0; \quad \frac{\partial_s \sigma_{xy}}{\partial x} + \frac{\partial_s \sigma_{yy}}{\partial y} - \frac{\partial_a \sigma_{yx}}{\partial x} = 0 \\ \frac{\partial m_{xz}}{\partial x} + \frac{\partial m_{yz}}{\partial y} + 2({}_a\sigma_{yx}) = 0\end{gathered} \tag{2.128}$$

$${}_s\sigma_{xx} = D_{11}\frac{\partial u}{\partial x} + D_{12}\frac{\partial u}{\partial y}; \quad {}_s\sigma_{yy} = D_{21}\frac{\partial u}{\partial x} + D_{22}\frac{\partial v}{\partial y}; \quad {}_s\sigma_{xy} = D_{33}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{2.129}$$

$$\begin{gathered}{}_s m_{xz} = \left(\frac{E_0}{m_0 L_0}\right)\underset{\sim}{\alpha}\frac{\partial({}_i\Theta_z)}{\partial x}; \quad {}_s m_{yz} = \left(\frac{E_0}{m_0 L_0}\right)\underset{\sim}{\alpha}\frac{\partial({}_i\Theta_z)}{\partial y} \\ {}_a m_{xz} = \left(\frac{E_0}{m_0 L_0}\right)\underset{\sim}{\beta}\frac{\partial({}_i\Theta_z)}{\partial x}; \quad {}_a m_{yz} = \left(\frac{E_0}{m_0 L_0}\right)\underset{\sim}{\beta}\frac{\partial({}_i\Theta_z)}{\partial y}\end{gathered} \tag{2.130}$$

$$\begin{gathered}D_{11} = D_{22} = \frac{E}{1-\nu^2}; \quad D_{12} = D_{21} = \frac{\nu E}{1-\nu^2}; \quad D_{33} = G = \frac{E}{2(1+\nu)} \\ {}_i\Theta_z = \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)\end{gathered} \tag{2.131}$$

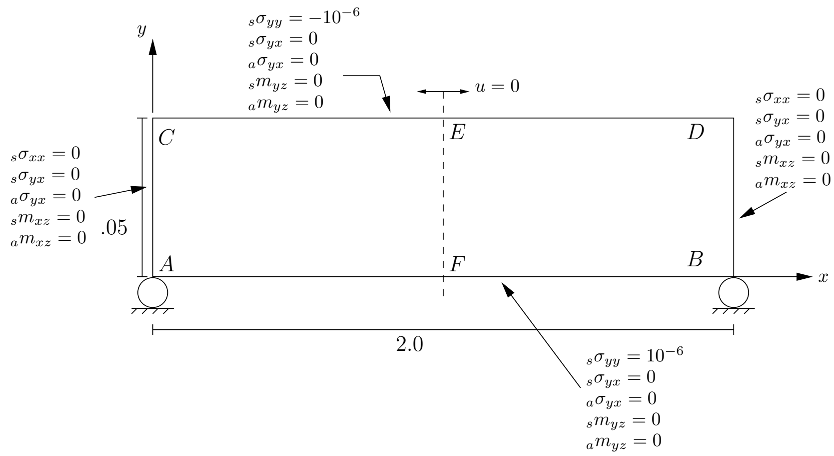
In (2.128)-(2.131), $\frac{E_0}{m_0 L_0}$ is in fact one, but it has been left in constitutive theory for the moment tensors for sake of clarity. Equations (2.128)-(2.131) are a system of eleven first order linear coupled differential equations in eleven dependent variables u, v, ${}_s\sigma_{xx}$, ${}_s\sigma_{yy}$, ${}_s\sigma_{xy}$, ${}_a\sigma_{yx}$, ${}_s m_{xz}$, ${}_s m_{yz}$, ${}_a m_{xz}$, ${}_a m_{yz}$ and ${}_i\Theta_z$.

2.8.2 Model problem details and results

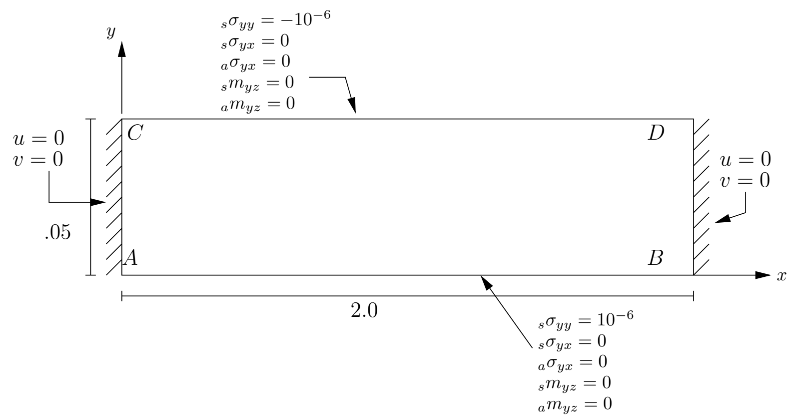
One considers a thin plate with length $\hat{l}$ of 20 inches, width $\hat{b}$ of 0.5 inches and thickness $\hat{t}$ of 0.1 inches. With $L_0 = 10$ inches the dimensionless plate is 2 x 0.05 x 0.01. One considers uniformly applied stress in the y direction on the top and bottom faces of the plate (in the plane of the plate). Choosing $\hat{E} = E_0 = 30$ x 10^6 psi, hence $E = 1$. Dimensionless $\underset{\sim}{\alpha} = \frac{\hat{\underset{\sim}{\alpha}}}{E_0}$ and $\underset{\sim}{\beta} = \frac{\hat{\underset{\sim}{\beta}}}{E_0}$ are varied to study their influence on the deformation physics. Clearly for $\underset{\sim}{\alpha} = 0$ and $\underset{\sim}{\beta} = 0$, the internal polar physics is absent i.e the usual small strain approximation theory of elasticity applies for this case.

2.8.2.1 Model problem 1

In this case one considers simply supported plate as shown in figure 2.1(a). Points A and B are constrained in the y direction, but are free to move in the x direction. On face AB of the plate $\sigma_{yy} = 10^{-6}$ and on face CD, $\sigma_{yy} = -10^{-6}$ is applied causing deflection of the plate in the negative y direction. At the center plane (EF) the x displacement is constrained (due to symmetry). Since $\hat{b}$ and $\hat{t}$ are much smaller than $\hat{l}$, the deformation behavior is like a simply supported 2D slender beam (shear deformation is not significant). The domain (l x b) 2 x 0.05 is discretized using a twenty element uniform discretization (ten elements along the length l and two elements along the width b) using nine node p-version hierarchical plane stress elements with higher order global differentiability local approximations in $H^{k,p}(\bar{\Omega}^e)$ scalar product spaces. Boundary conditions on the four boundaries of the domain $ABCD$ of the beam are also shown in figure 2.1(a). The nine node elements are mapped in a two unit square with the origin of the coordinate system ξ,η. The degrees of local approximation space in ξ and η (p_ξ, p_η) are chosen to be equal $p = p_\xi = p_\eta$ and are chosen to be the same for all dependent variables. Since the mathematical model is a system of first order differential equations, if the order of approximation space in x and y is chosen to be two i.e. local approximations of class C^1 in both x and y, then the integrals over the discretization are Riemann. On the other hand, if one choses the order of approximation to be one, then the local approximations are of class C^0 implying that the integrals over the discretization are in Lebesgue sense. Due to the smoothness of the solution of the model problem, both choices work well, i.e. the C^0 solutions approach C^1 solutions upon convergence, but in the weak sense. In the results presented here, $k = 1$ i.e. local approximations of class C^0 is chosen. A p-convergence study with $p = p_\xi = p_\eta = 3, 5, ..$ shows that at p=9 the integrated sum of squares of the residuals are



(a) Simply suppported plate



(b) Fixed-fixed plate

Figure 2.1: Schematics and boundary conditions for simply supported thin plate and fixed-fixed thin plate (dimensionless)

of the order of $O(10^{-16})$, confirming that the equations in the mathematical model are satisfied accurately in the pointwise sense. This is confirmed by the similar studies with solutions of class C^1 and their comparison with C^0 studies. Thus, one presents results for $p = p_\xi = p_\eta = 9$ with local approximations of class C^0 for all dependent variables using the 20 element uniform discretization described earlier.

2.8.2.2 Model problem 2

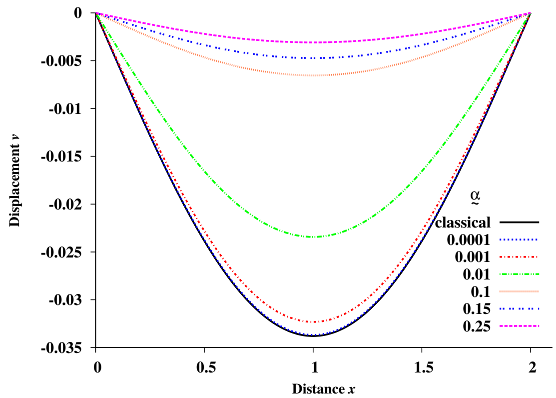
This model problem consists of the same plate as used in model problem 1 but is considered clamped at the two ends ($x = 0$ and $x = 2$ shown in figure 2.1 (b)). The boundary conditions on the boundaries AB and CD (excluding points A and B) remain the same as in model problem 1. Boundary conditions on AC and BD (clamped boundaries) are $u = v = 0$ as shown in figure 1(b). The details of the discretization, choice of order of approximation space etc. are the same for this model problem as those described for model problem 1. In this case also, a p-convergence study for solutions of class C^0 yields integrated sum of squares of the residual of the order of $O(10^-16)$ as in the case of model problem 1. Thus, for this problem also, $p = p_\xi = p_\eta = 9$ and C^0 local approximations for all dependent variables yields very accurate solutions, hence are used to compute the results presented here.

2.8.3 Solutions for model problem: 1 and 2

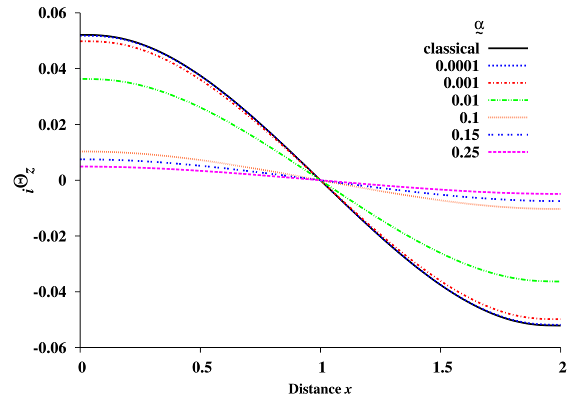
In the solutions presented here for model problems 1 and 2, the dimensionless modulus of elasticity E of 1 corresponds to $\hat{E} = 30$ x 10^{-16} psi. Poisson's ratio is chosen to be 0.3. Material coefficients $\underset{\sim}{\alpha}$ and $\underset{\sim}{\beta}$ related to the internal polar physics are chosen to vary between $0.0001 - 0.25$. As shown earlier in the theoretical details section, increasing values of $\underset{\sim}{\alpha}$ represent increasing presence of polar physics i.e.progressively increasing resistance to deformation whereas for a fixed values of $\underset{\sim}{\alpha}$, progressively increasing values of $\underset{\sim}{\beta}$ correspond to progressively diminishing internal polar physics and when $\underset{\sim}{\beta} = \underset{\sim}{\alpha}$ the internal polar physics is completely absent and the behavior reduces to classical continuum description.

2.8.3.1 Simply supported thin plate

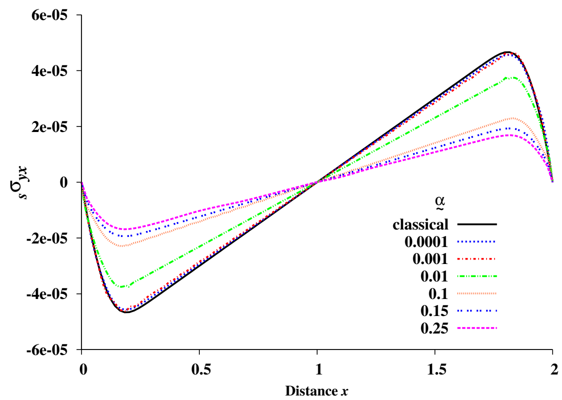
Figures 2.2 (a),(b),(c),(d),(e) show plots of v, ${}_i\Theta_z$, ${}_s\sigma_{yx}$, ${}_a\sigma_{yx}$, ${}_sm_{xz}$ versus x at $y = 0.025$ for $\underset{\sim}{\alpha} = 0.0001, 0.001, 0.01, 0.1, 0.15, 0.25$ and $\underset{\sim}{\beta} = 0.0$. For this case there is no constitutive theory for ${}_am_{xz}$, hence m_{xz}= ${}_sm_{xz}$+${}_am_{xz}$= ${}_sm_{xz}$ i.e. moment tensor is symmetric. Thus, these results correspond to the case when the balance of moments of moments is used as a balance law. From figure 2.2 one observes that progressively increasing values of $\underset{\sim}{\alpha}$ represent progressively increasing resistance to deformation thereby producing progressively reduced transverse displacement v, reduced rotation ${}_i\Theta_z$, reduced ${}_s\sigma_{yx}$, increased ${}_a\sigma_{yx}$ and increased ${}_sm_{xz} = m_{xz}$ along the length



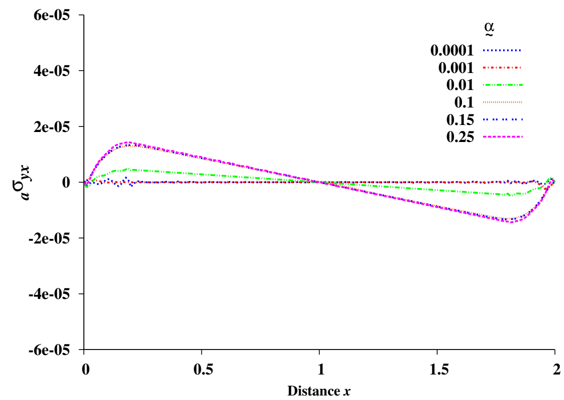
(a) v versus x at $y = 0.025$



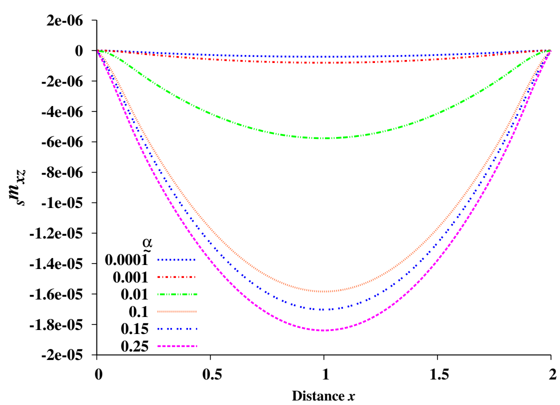
(b) $_i\Theta_z$ versus x at $y = 0.025$



(c) $_s\sigma_{yx}$ versus x at $y = 0.025$



(d) $_a\sigma_{yx}$ versus x at $y = 0.025$



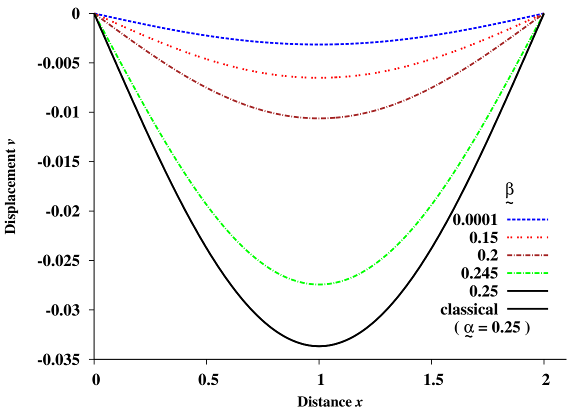
(e) $_sm_{xz}$ versus x at $y = 0.025$

Figure 2.2: Graphs of v, $_i\Theta_z$, $_s\sigma_{yx}$, $_a\sigma_{yx}$, $_sm_{xz}$ versus x at $y = 0.025$ with varying $\underset{\sim}{\alpha}$ and $\underset{\sim}{\beta} = 0$ for simply supported thin plate

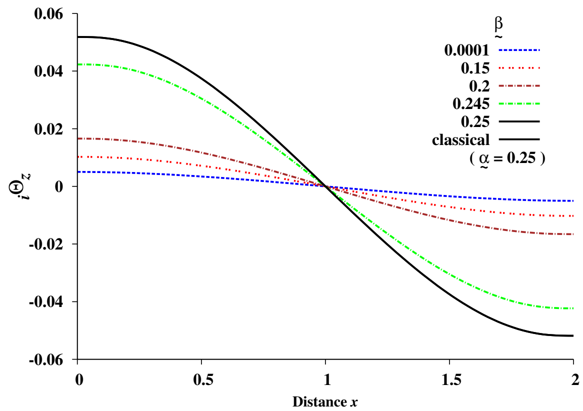
of the beam (at $y = 0.025$). This behavior is consistent with the physics due to internal rotations arising from the antisymmetric part of the displacement gradient tensor.

In the next study one assumes that balance of moments of moments is not a balance law, hence $\boldsymbol{m}$ is non-symmetric and one has constitutive theory for ${}_s\boldsymbol{m}$ and ${}_a\boldsymbol{m}$. To illustrate the adverse influence on the deformation physics of the constitutive theory for ${}_a\boldsymbol{m}$, one begins by considering presence of some polar physics due to a chosen non-zero value of $\underset{\sim}{\alpha}$. One chooses $\underset{\sim}{\alpha} = 0.25$ and perform computations for progressively increasing $\underset{\sim}{\beta}$ while keeping $\underset{\sim}{\alpha} = 0.25$ and monitor resulting behavior of v, ${}_i\Theta_z$, ${}_s\sigma_{yx}$, ${}_a\sigma_{yx}$, ${}_sm_{xz}$, ${}_am_{xz}$ and m_{xz}. In the calculations values of $\underset{\sim}{\beta}$ = $0.0001, 0.15, 0.2, 0.245, 0.25$ for fixed $\underset{\sim}{\alpha} = 0.25$ is considered. Figure 2.3(a)-(d) show plots of v, ${}_i\Theta_z$, ${}_s\sigma_{xy}$, ${}_a\sigma_{yx}$ and figure 2.4(a)-(b) shows plot of ${}_sm_{xz}$ and ${}_am_{xz}$ versus x at $y = 0.025$ along the length of the beam for $\underset{\sim}{\alpha} = 0.25$ and progressively increasing values of $\underset{\sim}{\beta}$. Figure 2.4(c) shows plot of ${}_sm_{xz}$, ${}_am_{xz}$ versus x at $y = 0.025$ for $\underset{\sim}{\alpha} = \underset{\sim}{\beta} = 0.0001 - 0.25$. For $\underset{\sim}{\alpha} = 0.25$ and $\underset{\sim}{\beta} = 0.0001$ the behaviors of v, ${}_i\Theta_z$, ${}_s\sigma_{yx}$, ${}_a\sigma_{yx}$ in figures (a)-(d) are same as those shown in figure 2.2 (correct internal polar physics corresponding to $\underset{\sim}{\beta} = 0.0$). With progressively increasing values of $\underset{\sim}{\beta}$, v, ${}_i\Theta_z$, ${}_s\sigma_{yx}$ (figures 2.3(a)-(c)) progressively increase while ${}_a\sigma_{yx}$, ${}_sm_{xz}$, ${}_am_{xz}$ progressively decrease and at $\underset{\sim}{\beta} = 0.25$ one has $\underset{\sim}{\alpha} - \underset{\sim}{\beta} = 0$, hence one recovers classical continuum physics i.e. the internal polar physics completely disappears. The study demonstrates that the internal polar physics (resistance to deformation) present at $\underset{\sim}{\alpha} = 0.25$ begins to diminish as $\underset{\sim}{\beta}$ increases and when $\underset{\sim}{\beta} = \underset{\sim}{\alpha} = 0.25$ the internal polar physics is not present at all. From figure 2.4(c) one note that when $\underset{\sim}{\alpha} = \underset{\sim}{\beta}$, regardless of the value of $\underset{\sim}{\alpha}$ (or $\underset{\sim}{\beta}$), ${}_am_{xz} = -{}_sm_{xz}$ holds. This clearly demonstrates that the presence of the constitutive theory for ${}_a\boldsymbol{m}$ is non physical as it removes existence of polar physics due to $\underset{\sim}{\alpha}$.

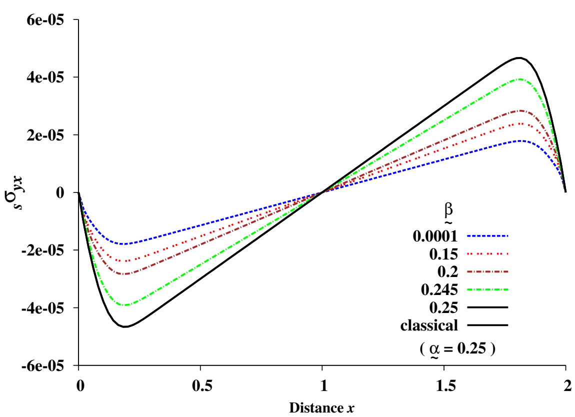
In the third study $\underset{\sim}{\alpha} = 0$ and non-zero values of $\underset{\sim}{\beta}$ is chosen. This of course implies that Cauchy moment tensor is purely anti-symmetric. It is already shown in the theoretical development section that this is not possible as it is not supported by energy equation and entropy inequality. Nonetheless one presents numerical studies here to demonstrate serious adverse consequences of doing so. One considers $\underset{\sim}{\alpha} = 0$ and vary $\underset{\sim}{\beta}$ between 0.0001 and 0.016 (shown in graphs in figure 2.5). At



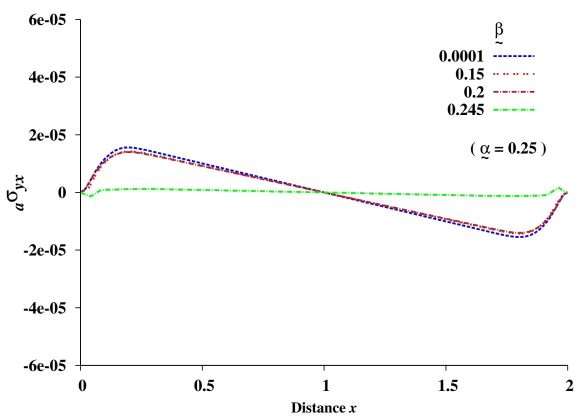
(a) v versus x at $y = 0.025$



(b) $_i\Theta_z$ versus x at $y = 0.025$



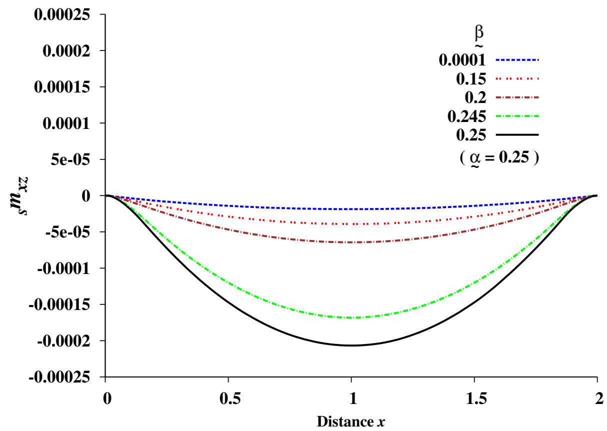
(c) $_s\sigma_{yx}$ versus x at $y = 0.025$



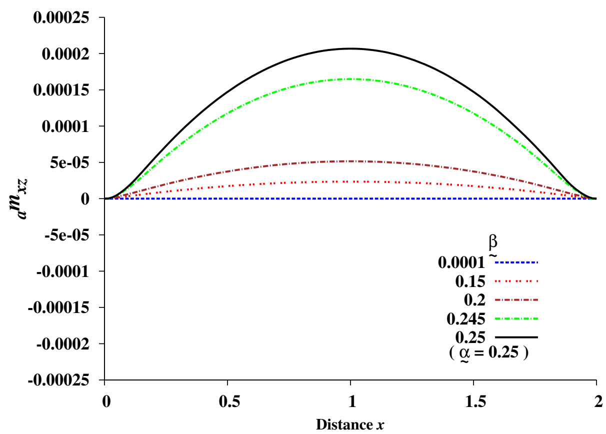
(d) $_a\sigma_{yx}$ versus x at $y = 0.025$

Figure 2.3: Graphs of v, $_i\Theta_z$, $_s\sigma_{yx}$, $_a\sigma_{yx}$ versus x at $y = 0.025$ with $\underset{\sim}{\alpha} = 0.25$ and varying $\underset{\sim}{\beta}$ for simply supported thin plate

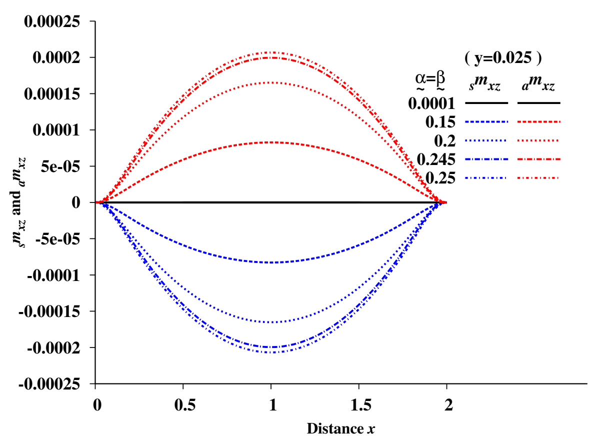
$\underset{\sim}{\alpha} = 0$ and $\underset{\sim}{\beta} = 0.0001$ one almost has classical continuum theory behaviors for v, $_i\Theta_z$ and $_s\sigma_{yx}$. Zero values of $_a\sigma_{yx}$ and $\boldsymbol{m}$ (shown in the graphs in figure 2.5) confirm classical continuum behavior. With progressively increasing $\underset{\sim}{\beta}$ one observes progressively increasing values of v, $_i\Theta_z$ (versus x at $y = 0.025$) that imply progressively decreasing ability of the matter to resist deformation. This is exactly opposite of what is expected due to the presence of internal polar physics. Behavior of $_s\sigma_{yx}$, $_a\sigma_{yx}$ and m_{xz} in figures 2.5 (c),(d),(e) are consistent with behavior of v and $_i\Theta_z$ in figures 2.5



(a) ${}_sm_{xz}$ versus x at $y = 0.025$

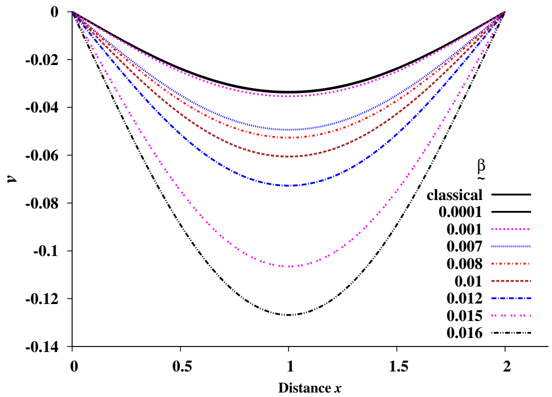


(b) ${}_am_{xz}$ versus x at $y = 0.025$

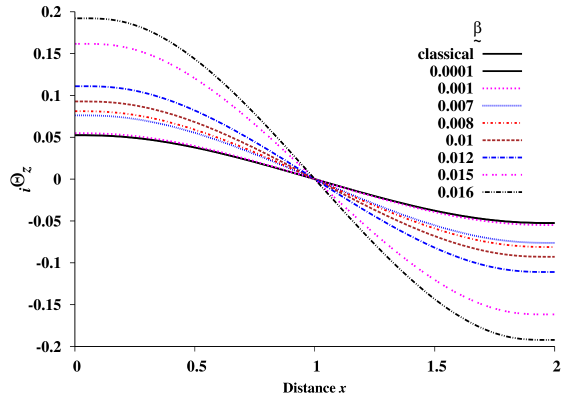


(c) ${}_sm_{xz}$ and ${}_am_{xz}$ versus x at $y = 0.025$

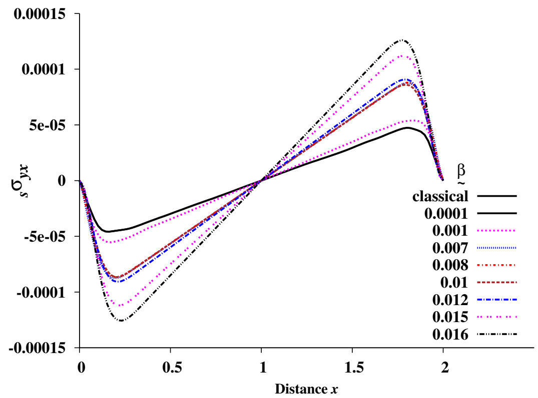
Figure 2.4: Graphs of ${}_sm_{xz}$, ${}_am_{xz}$ versus x at $y = 0.025$ with $\underset{\sim}{\alpha} = 0.25$ and varying $\underset{\sim}{\beta}$ for simply supported thin plate



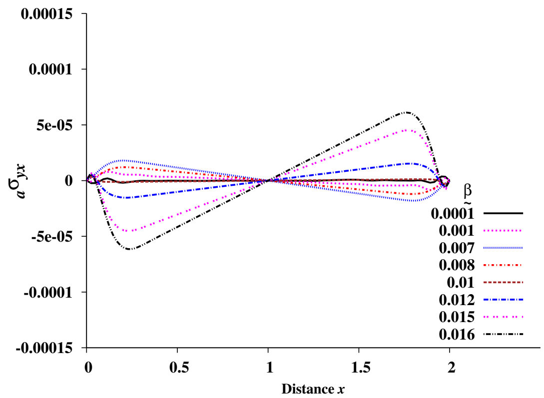
(a) v versus x at $y = 0.025$



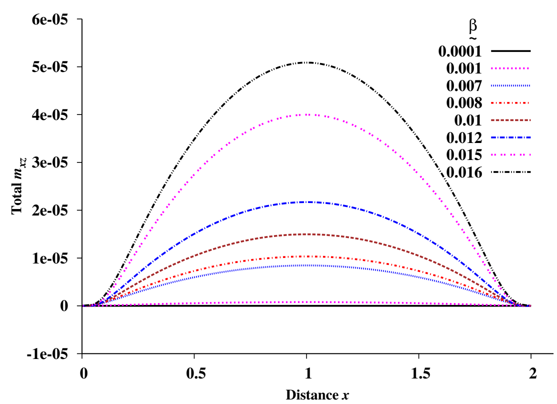
(b) $_i\Theta_z$ versus x at $y = 0.025$



(c) $_s\sigma_{yx}$ versus x at $y = 0.025$



(d) $_a\sigma_{yx}$ versus x at $y = 0.025$



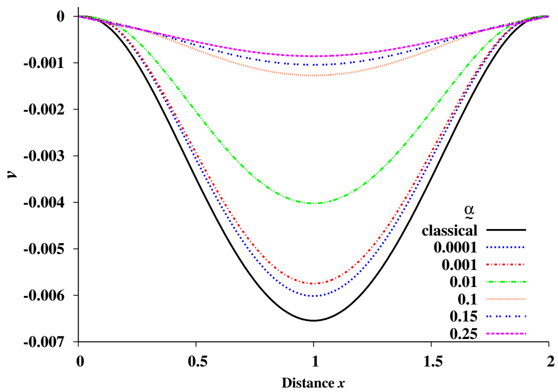
(e) Total m_{xz}versus x at $y = 0.025$

Figure 2.5: Graphs of v, $_i\Theta_z$, $_s\sigma_{yx}$, $_a\sigma_{yx}$, m_{xz} versus x at $y = 0.025$ with varying $\underset{\sim}{\beta}$ for simply supported thin plate

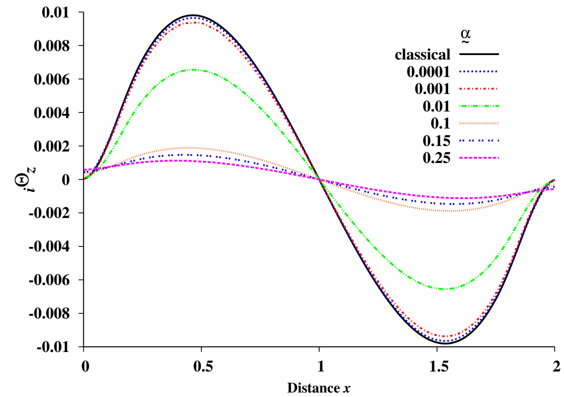
(a),(b). This study demonstrates that the constitutive theory for ${}_am_{xz}$ produces behavior exactly opposite to internal polar physics i.e. the presence of constitutive theory for ${}_a\boldsymbol{m}$ results in reduced stiffness in the deforming matter implying progressively decreasing ability of the deforming matter to resist load. This behavior is precisely opposite of what is expected.

2.8.3.2 Fixed-fixed thin plate

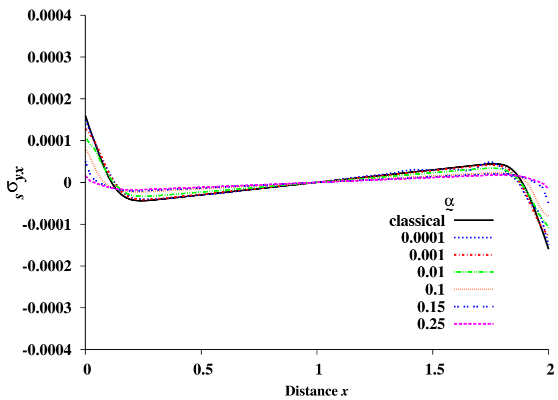
Results for fixed-fixed case show presence of correct internal polar physics with progressively increasing values of $\underset{\sim}{\alpha}$ with $\underset{\sim}{\beta} = 0$ (figure 2.6). When $\underset{\sim}{\alpha} = 0.25$ (fixed) and $\underset{\sim}{\beta}$ is progressively increased, one observes that progressively reducing polar physics with increasing $\underset{\sim}{\beta}$ and when $\underset{\sim}{\alpha}$ = $\underset{\sim}{\beta}$ = 0.25, polar physics is completely absent (figure 2.7 and figure 2.8). One notes in figure 2.8(c) that when $\underset{\sim}{\alpha}$ = $\underset{\sim}{\beta}$, regardless of the value of $\underset{\sim}{\alpha}$ (or $\underset{\sim}{\beta}$), ${}_am_{xz} = -{}_sm_{xz}$ holds, confirming the nonphysical nature of the constitutive theory for ${}_a\boldsymbol{m}$. When $\underset{\sim}{\alpha} = 0$ and $\underset{\sim}{\beta}$ is progressively increased, one observes softening behavior of the matter compared to classical theory resulting in increasing v, ${}_i\Theta_z$ compared to classical theory which of course is contrary to the internal polar physics (figure 2.9).



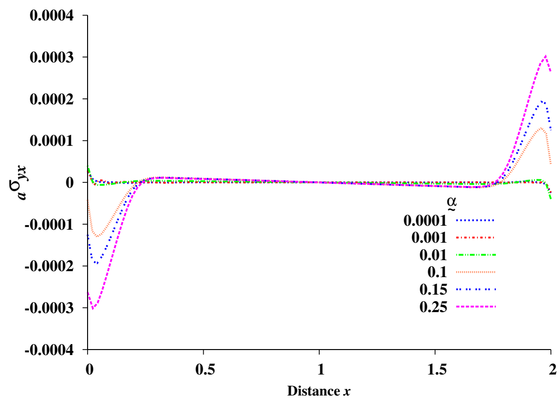
(a) v versus x at $y = 0.025$



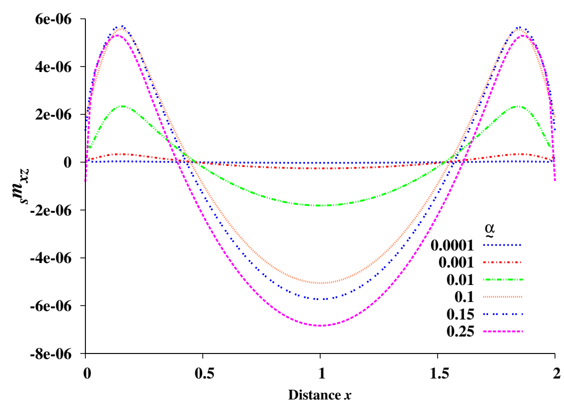
(b) ${}_{i}\Theta_{z}$ versus x at $y = 0.025$



(c) ${}_{s}\sigma_{yx}$ versus x at $y = 0.025$

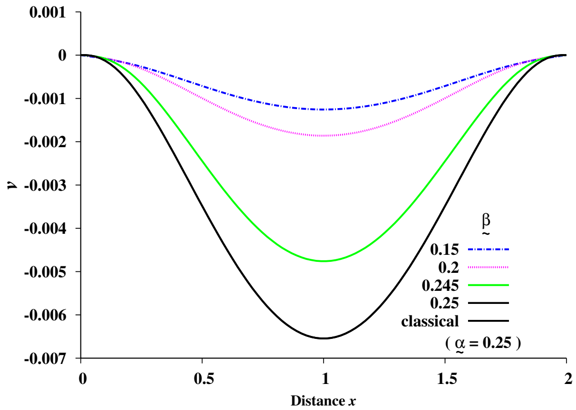


(d) ${}_{a}\sigma_{yx}$ versus x at $y = 0.025$

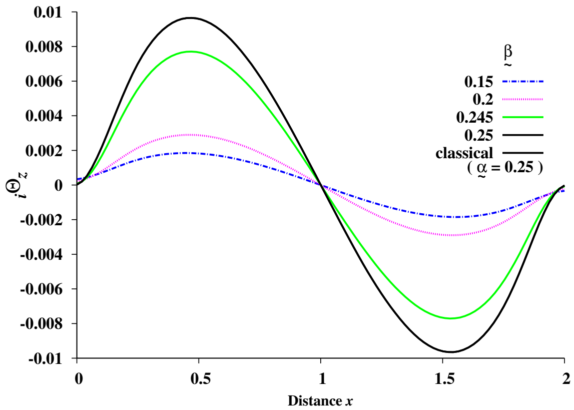


(e) ${}_{s}m_{xz}$ versus x at $y = 0.025$

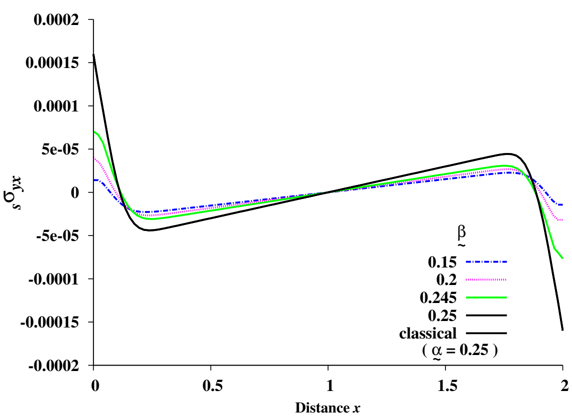
Figure 2.6: Graphs of v, ${}_{i}\Theta_{z}$, ${}_{s}\sigma_{yx}$, ${}_{a}\sigma_{yx}$, ${}_{s}m_{xz}$ versus x at $y = 0.025$ with varying $\underset{\sim}{\alpha}$ and $\underset{\sim}{\beta} = 0$ for fixed-fixed thin plate



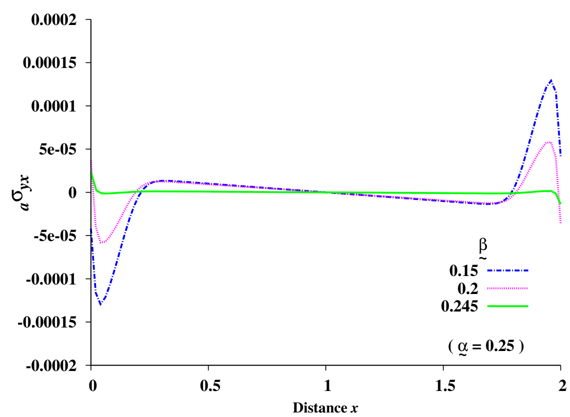
(a) v versus x at $y = 0.025$



(b) ${}_i\Theta_z$ versus x at $y = 0.025$

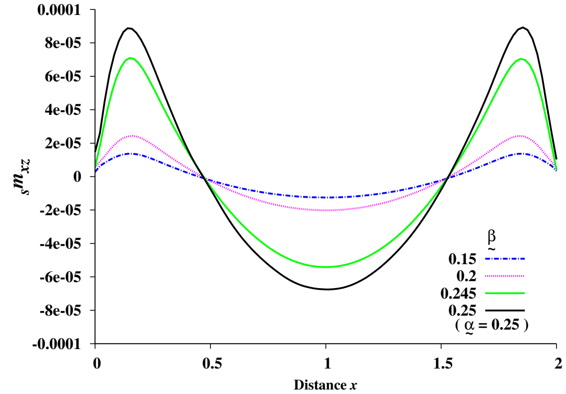


(c) ${}_s\sigma_{yx}$ versus x at $y = 0.025$

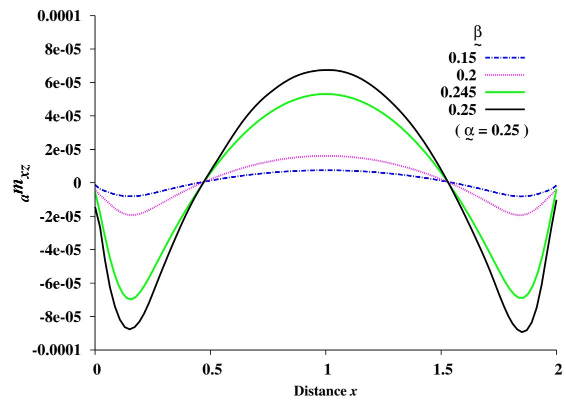


(d) ${}_a\sigma_{yx}$ versus x at $y = 0.025$

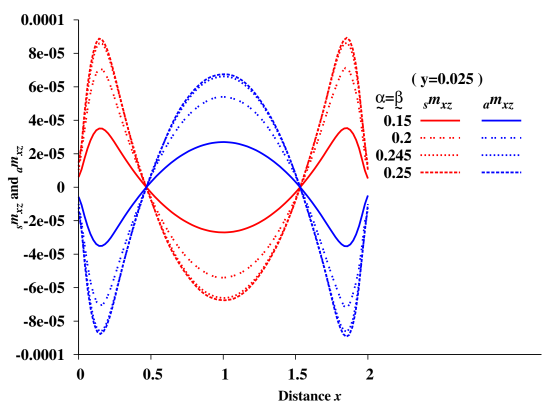
Figure 2.7: Graphs of v, ${}_i\Theta_z$, ${}_s\sigma_{yx}$, ${}_a\sigma_{yx}$ versus x at $y = 0.025$ with $\underset{\sim}{\alpha} = 0.25$ and varying $\underset{\sim}{\beta}$ for fixed-fixed thin plate



(a) $_sm_{xz}$ versus x at $y = 0.025$

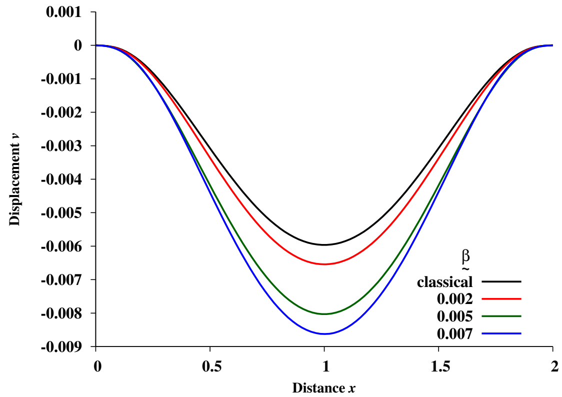


(b) $_am_{xz}$ versus x at $y = 0.025$

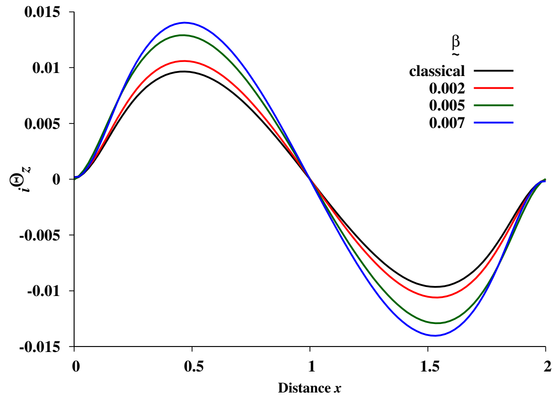


(c) $_sm_{xz}$ and $_am_{xz}$ versus x at $y = 0.025$

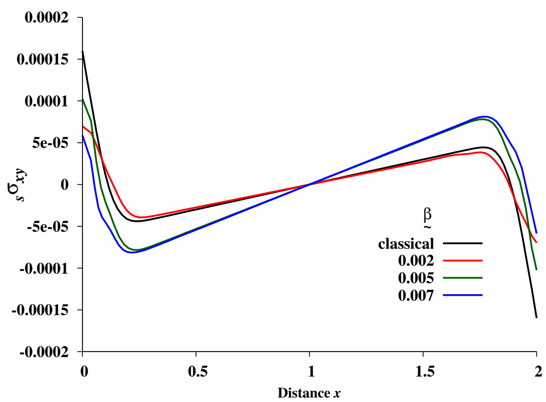
Figure 2.8: Graphs of $_sm_{xz}$, $_am_{xz}$ versus x at $y = 0.025$ with $\underset{\sim}{\alpha} = 0.25$ and varying $\underset{\sim}{\beta}$ for fixed-fixed thin plate



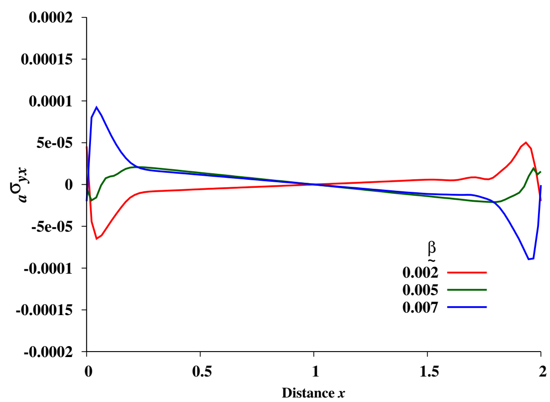
(a) v versus x at $y = 0.025$



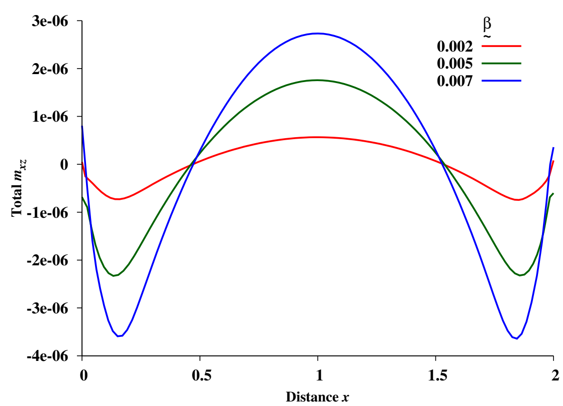
(b) $_i\Theta_z$ versus x at $y = 0.025$



(c) $_s\sigma_{xy}$ versus x at $y = 0.025$



(d) $_a\sigma_{yx}$ versus x at $y = 0.025$



(e) Total m_{xz}versus x at $y = 0.025$

Figure 2.9: Graphs of v, $_i\Theta_z$, $_s\sigma_{xy}$, $_a\sigma_{yx}$, m_{xz} versus x at $y = 0.025$ with varying $\underset{\sim}{\beta}$ for fixed-fixed thin plate

2.9 Conclusions

The necessity of the "balance of moments of moments" balance law for non-classical continuum theories for solids as an additional balance law compared to classical continuum theory is demonstrated from the theoretical considerations as well as through model problem studies. It has been shown that in the absence of balance of moments of moments balance law, the Cauchy moment tensor is not symmetric. A decomposition of Cauchy moment tensor into symmetric (${}_s\boldsymbol{m}$) and anti-symmetric tensors (${}_a\boldsymbol{m}$) and their substitution in the entropy inequality shows through conjugate pairs that constitutive theories for both ${}_s\boldsymbol{m}$ and ${}_a\boldsymbol{m}$ are necessary. On the other hand when the balance of moments of moments is a balance law the Cauchy moment tensor is symmetric ($\boldsymbol{m} = {}_s\boldsymbol{m}$) and ${}_a\boldsymbol{m} = 0$; hence only the constitutive theory for ($\boldsymbol{m} = {}_s\boldsymbol{m}$) is necessary. One notes the following based on the work presented here.

1. Using linear constitutive theories for ${}_s\boldsymbol{m}$ and ${}_a\boldsymbol{m}$ it is shown that these lead to $\boldsymbol{m} = \boldsymbol{m}({}^{\Theta}\boldsymbol{J}, \theta)$ with the assumption that the material coefficients $\underset{\sim}{\alpha}$ and $\underset{\sim}{\beta}$ in the constitutive theories for ${}_s\boldsymbol{m}$ and ${}_a\boldsymbol{m}$ are same. Since $\boldsymbol{m}$ and ${}^{\Theta}\boldsymbol{J}$ both are non-symmetric tensors, based on the works of Spencer, Wang and Zheng etc. [52–68] $\boldsymbol{m} = \boldsymbol{m}({}^{\Theta}\boldsymbol{J}, \theta)$ is not possible i.e. generators of $\boldsymbol{m}$ can not be determined using ${}^{\Theta}\boldsymbol{J}$.

2. Using 2-D plane non-polar elasticity (plane stress) and choosing $\underset{\sim}{\alpha} \neq \underset{\sim}{\beta}$, the constitutive theory for ${}_a m$ produces exactly opposite behavior to that of the constitutive theory for ${}_s m$. When $\underset{\sim}{\beta} = 0$ i.e. no constitutive theory for ${}_a\boldsymbol{m}$ (hence presence of balance of moments of moments as a balance law) one observes that increasing presence of polar physics (resistance to deformation) with progressively increasing $\underset{\sim}{\alpha}$. This is exactly what is the expected behavior due to internal polar physics. For non-zero value of $\underset{\sim}{\alpha}$ (0.25 used in numerical studies) i.e. presence of some polar physics, one note thats progressively increasing values of $\underset{\sim}{\beta}$ result in progressively diminishing polar physics initially present when $\underset{\sim}{\beta} = 0$ and when $\underset{\sim}{\beta} = \underset{\sim}{\alpha}$ we recover classical continuum behavior. When $\underset{\sim}{\alpha} = 0$ and $\underset{\sim}{\beta} \neq 0$, progressively softening behavior of the solid matter is observed for progressively increasing values of $\underset{\sim}{\beta}$, this of course

is non-physical. These aspects have been demonstrated through model problems also.

3. From both (1) and (2), one can conclude that the presence of constitutive theory for ${}_a\boldsymbol{m}$ is neither justified based on theoretical consideration (that leads to $\boldsymbol{m} = \boldsymbol{m}({}^{\Theta}J, \theta)$, which is not possible) nor based on the observed behaviors in the model problem studies. Elimination of the constitutive theory for ${}_a\boldsymbol{m}$ requires that balance of moments of moments be an additional balance law in the non-classical continuum theories as this additional balance law restricts Cauchy moment tensor to be symmetric thereby eliminating ${}_a\boldsymbol{m}$ all together.

4. Additional arguments in favor of the balance of moments of moments balance law for non-classical continuum theories have also been presented. These basically point to the fact that presence of internal rotations and conjugate moment tensor is additional physics over and beyond classical continuum mechanics, hence it is reasonable to assume that the conservation and balance laws that are necessary for classical continuum mechanics may not be sufficient in case of non-classical continuum mechanics to ensure equilibrium of the deforming matter. Inductive reasoning is presented to demonstrate that additional new kinematic variable in the description of the physics would necessitate an additional balance law that would ensure equilibrium of the deforming matter.

5. It is rather clear that constitutive theory for ${}_a\boldsymbol{m}$ only (i.e. absence of ${}_s\boldsymbol{m}$) has absolutely no basis regardless of whether one uses balance of moments of moments as a balance law or not.

In conclusion the work presented in this chapter establishes through theoretical developments and through model problem studies that the balance of moments of moments is an essential balance law in non-classical continuum theories to: 1) incorporate correct physics of internal rotations and conjugate moments. 2) ensure that the deforming matter is in equilibrium during the deformation process.

Chapter 3

Finite Element Processes Based on GM/WF in Non-Classical Solid Mechanics

In non-classical thermoelastic solids incorporating internal rotation and conjugate Cauchy moment tensor the mechanical deformation is reversible. This suggests that within the realm of linear mathematical models that only consider small strains and small deformation the mechanical deformation is reversible. Hence, it is possible to recast the conservation and balance laws along with constitutive theories in a form that adjoint $\boldsymbol{A}^*$ of the differential operator $\boldsymbol{A}$ in mathematical model is same as the differential operator $\boldsymbol{A}$. This holds regardless of whether one considers an intial value problem (IVP) (when the integrals over open boundary are neglected) or boundary value problem (BVP). Thus, in such cases Galerkin method with weak form (GM/WF) for BVPs and space-time Galerkin method with weak form (STGM/WF) for IVPs are highly meritorious due to the fact that: 1) the integral form for BVPs are variationally consistent (VC) and 2) the space-time integral forms for IVP are space time variationally consistent (STVC). The consequence of VC and STVC integral forms is that the resulting coefficient matrices are symmetric and positive definite ensuring unconditionally stable computational processes for both BVPs and IVPs. Other benefits of GM/WF and space-time GM/WF are ease of specifying boundary conditions and initial conditions specially traction boundary conditions and initial conditions on curved boundaries. While VC and STVC feature also exists in least squares process (LSP) and space-time least squares finite element processes (STLSP) for BVPs and IVPs, the ease of specifying traction boundary conditions feature in GM/WF and STGM/WF is highly meritorious compared to LSP and STLSP.

A serious disadvantage of GM/WF and STGM/WF is that the mathematical models (momentum equations) needed in the desired form contain higher order derivatives of displacements (upto fourth order), hence necessitate use of higher order spaces in their solution. As well known, this problem can be easily overcome in LSP and STLSP by introduction of auxillary equations and auxillary variables, thus keeping the highest orders of the derivatives of the dependent variables to one or any other desired order. A serious disadvatange of this approach in LSP is the significant increase in the number of dependent variables, hence poor computational efficiency. In this chapter one considers non-classical continuum models for internally polar linear elastic solids in which internal rotations due to displacement gradient tensor (hence internal polar physics) are considered in the conservation and the balance laws and the constitutive theories. For simplicity, one only considers isothermal case, hence energy equation is not part of mathematical model. When using mathematical models derived in displacements are used in constructing integral form using GM/WF, the number of dependent variables are reduced drastically (only three in $\mathbb{R}^3$) whereas in case of first order systems used in LSP and STLSP one may have as many as 22 for isothermal case. Thus, GM/WF results in dramatic improvement in computational efficiency as well as accuracy when minimally conforming spaces are used for approximation. In this chapter one only considers mathematical model in $\mathbb{R}^2$ for BVPs (for simplicity). The integral form is derived in $\mathbb{R}^2$ using GM/WF. Numerical examples are presented to demonstrate advantages of finite element process derived using integral form based on GM/WF for non-classical linear theories for solids incorporating internal rotations due to displacement gradient tensor.

3.1 Mathematical Model

For non-classical elastic solid matter with internal rotation and conjugate moment physics undergoing small deformation and small strain, the mathematical model for BVPs has been presented by Surana et al. [3–5, 69–71]. In present work one assumes isothermal deformation process i.e. no entropy production due to mechanical work, hence the mathematical model in Lagrangian description consists of balance of linear momenta, balance of angular momenta, balance of moments of moments (as a balance law or its absence) [3–5, 69–72] and the constitutive theories for: symmetric part of Cauchy stress tensor, symmetric part of Cauchy moment tensor and antisymmetric part of Cauchy moment tensor (if balance of moments of moments is not used as a balance law). We have the following dimensionless form of the mathematical model in $\mathbb{R}^2$ (neglecting body forces) assuming that balance of moments of moments is not a balance law [72]. Using the decomposition of the Cauchy moment tensor into symmetric and antisymmetric tensors $\boldsymbol{m} =_s \boldsymbol{m} + {}_a\boldsymbol{m}$, constitutive theories for ${}_s\boldsymbol{m}$ and ${}_a\boldsymbol{m}$ is obtained. Choosing $x = x_1$, $y = x_2$, $u = u_1$, $v = u_2$ one can write the following for balance laws and the constitutive theories $\forall\ x, y \in \Omega_{xy}$.

$$\frac{\partial_s\sigma_{xx}}{\partial x} + \frac{\partial_s\sigma_{yx}}{\partial y} + \frac{\partial_a\sigma_{yx}}{\partial y} = 0 \tag{3.1}$$

$$\frac{\partial_s\sigma_{xy}}{\partial x} + \frac{\partial_s\sigma_{yy}}{\partial y} - \frac{\partial_a\sigma_{yx}}{\partial x} = 0 \tag{3.2}$$

$$\begin{gathered}\frac{\partial m_{xz}}{\partial x} + \frac{\partial m_{yz}}{\partial y} + 2({}_a\sigma_{yx}) = 0 \\ m_{xz} = {}_sm_{xz} + {}_am_{xz} \\ m_{yz} = {}_sm_{yz} + {}_am_{yz}\end{gathered} \tag{3.3}$$

$${}_s\sigma_{xx} = D_{11}\frac{\partial u}{\partial x} + D_{12}\frac{\partial u}{\partial y}; \quad {}_s\sigma_{yy} = D_{21}\frac{\partial u}{\partial x} + D_{22}\frac{\partial v}{\partial y}; \quad {}_s\sigma_{xy} = D_{33}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{3.4}$$

$${}_sm_{xz} = \left(\frac{E_0}{m_0 L_0}\right)\underset{\sim}{\alpha}\frac{\partial({}_i\Theta_z)}{\partial x} \tag{3.5}$$

$${}_sm_{yz} = \left(\frac{E_0}{m_0 L_0}\right)\underset{\sim}{\alpha}\frac{\partial({}_i\Theta_z)}{\partial y} \tag{3.6}$$

$$ {}_a m_{xz} = -\left(\frac{E_0}{m_0 L_0}\right) \underset{\sim}{\beta} \frac{\partial({}_i\Theta_z)}{\partial x} \tag{3.7} $$

$$ {}_a m_{yz} = -\left(\frac{E_0}{m_0 L_0}\right) \underset{\sim}{\beta} \frac{\partial({}_i\Theta_z)}{\partial y} \tag{3.8} $$

$$ {}_i\Theta_z = \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) \tag{3.9} $$

$$ D_{11} = D_{22} = \frac{E}{1-\nu^2}; \quad D_{12} = D_{21} = \frac{\nu E}{1-\nu^2}; \quad D_{33} = G = \frac{E}{2(1+\nu)} \tag{3.10} $$

The Cauchy stress tensor has also been decomposed into symmetric and antisymmetric tensors. In order to obtain the dimensionless equations (3.1)-(3.10), one can first write these with hat ($\hat{}$) on all quantities and variables indicating that they all have their usual dimensions in terms of length ($\hat{L}$), force ($\hat{F}$), and time ($\hat{t}$). If one chooses L_0, F_0 and t_0 as reference length, force and time, then the dimensionless length, force and time (L, F and t) are defined as

$$ L = \frac{\hat{L}}{L_0}, \quad F = \frac{\hat{F}}{F_0}, \quad t = \frac{\hat{t}}{t_0} \tag{3.11} $$

If one considers $\hat{E} = EE_0$, $\hat{x} = xL_0$, $\hat{y} = yL_0$, $\hat{m} = mm_0$, m_0=$\frac{\tau_0}{L_0}$, F_0=$\frac{\tau_0}{L_0}$, $\hat{\underset{\sim}{\alpha}}$=$\underset{\sim}{\alpha}E_0$, $\hat{\underset{\sim}{\beta}}$=$\underset{\sim}{\beta}E_0$ and choose L_0, E_0, then one obtains the dimensionless form of equations (3.1)-(3.10). In these $\frac{E_0}{m_0 L_0}$ is in fact unity but has been left in the constitutive theories for the moment tensor for sake of clarity.

Equations (3.1)-(3.9) are a system of eleven partial differential equations in eleven dependent variables u, v, ${}_s\sigma_{xx}$, ${}_s\sigma_{yy}$, ${}_s\sigma_{xy}$, ${}_a\sigma_{yx}$, ${}_s m_{xz}$, ${}_s m_{yz}$, ${}_a m_{xz}$, ${}_a m_{yz}$ and ${}_i\Theta_z$. One substitutes ${}_a\sigma_{yx}$ from (3.3) into (3.1) and (3.2).

$$ \frac{\partial({}_s\sigma_{xx})}{\partial x} + \frac{\partial({}_s\sigma_{yx})}{\partial y} + \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial(m_{xz})}{\partial x} + \frac{\partial(m_{yz})}{\partial y}\right) = A_1(u,v) = 0 \qquad \forall\, x, y \in \Omega_{xy} \tag{3.12} $$

$$ \frac{\partial({}_s\sigma_{xy})}{\partial x} + \frac{\partial({}_s\sigma_{yy})}{\partial y} - \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial(m_{xz})}{\partial x} + \frac{\partial(m_{yz})}{\partial y}\right) = A_2(u,v) = 0 \qquad \forall\, x, y \in \Omega_{xy} \tag{3.13} $$

3.2 Finite element formulation

The symmetric stress tensor components are defined by (3.4) and are function of u and v. Likewise m_{xz} and m_{yz} consists of the decomposition in (3.3) and the constitutive theories for symmetric and antisymmetric Cauchy moment tensors are given in (3.5)-(3.8) with ${}_i\Theta_z$ defined by (3.9), hence there are also functions of the gradients of u and v. Let $\bar{\Omega}^T_{xy} = \bigcup_e \bar{\Omega}^e_{xy}$ be the discretization of $\bar{\Omega}_{xy}$, domain of definition of mathematical model in which $\bar{\Omega}^e_{xy} = \Omega^e_{xy} \cup \Gamma^e$ is a typical finite element. A_1 and A_2 in (3.12) and (3.13) are differential operators that act on u and v (m_{yz} and m_{xz} are functions of the gradients of u and v and so are stresses). The balance of linear momenta equations in the form (3.12) and (3.13) are helpful in keeping the derivation of integral form based on GM/WF compact.

Let

$$\begin{aligned} w_1 &= \delta u_h \\ w_2 &= \delta v_h \end{aligned} \tag{3.14}$$

in which u_h and v_h are approximations of u and v over $\bar{\Omega}^T_{xy}$. Then, based on fundamental Lemma of the calculus of variation [73–87] one can construct scalar products of (3.1) and (3.2) with test functions w_1 and w_2 over the discretization $\bar{\Omega}^T_{xy}$

$$\begin{aligned} (A_1(u_h, v_h), w_1)_{\bar{\Omega}^T_{xy}} &= 0 \; ; \qquad w_1 = \delta u_h \\ (A_2(u_h, v_h), w_2)_{\bar{\Omega}^T_{xy}} &= 0 \; ; \qquad w_2 = \delta v_h \end{aligned} \tag{3.15}$$

or

$$\begin{aligned} (A_1(u_h, v_h), w_1) &= \sum_e (A_1(u^e_h, v^e_h), w_1)_{\bar{\Omega}^e_{xy}} = 0 \\ (A_2(u_h, v_h), w_2) &= \sum_e (A_2(u^e_h, v^e_h), w_2)_{\bar{\Omega}^e_{xy}} = 0 \end{aligned} \tag{3.16}$$

In which $u_h = \cup_e u^e_h$; $v_h = \cup_e v^e_h$ and u^e_h, v^e_h are local approximations of u and v over an element e

with domain $\bar{\Omega}^e_{xy}$. One considers $(A_1(u^e_h, v^e_h), w_1)_{\bar{\Omega}^e_{xy}}$ and $(A_2(u^e_h, v^e_h), w_2)_{\bar{\Omega}^e_{xy}}$ in which $w_1 = \delta u^e_h$, $w_2 = \delta v^e_h$ and substitute A_1 and A_2 from (3.12) and (3.13)

$$(A_1(u^e_h, v^e_h), w_1)_{\bar{\Omega}^e_{xy}} = \left(\frac{\partial({}_s\sigma_{xx})}{\partial x} + \frac{\partial({}_s\sigma_{yx})}{\partial y} - \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial(m_{xz})}{\partial x} + \frac{\partial(m_{yz})}{\partial y}\right), w1 \right)_{\bar{\Omega}^e_{xy}} \tag{3.17}$$

$$(A_2(u^e_h, v^e_h), w_2)_{\bar{\Omega}^e_{xy}} = \left(\frac{\partial({}_s\sigma_{xy})}{\partial x} + \frac{\partial({}_s\sigma_{yy})}{\partial y} + \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial(m_{xz})}{\partial x} + \frac{\partial(m_{yz})}{\partial y}\right), w_2 \right)_{\bar{\Omega}^e_{xy}} \tag{3.18}$$

Integration by parts once for each term in (3.17) and (3.18) yields and noting that $\bar{\Omega}^e_{xy} = \bar{\Omega}^e_{xy} \cup \Gamma^e, \Gamma^e$ being closed boundary of $\bar{\Omega}^e_{xy}$.

$$\begin{aligned}(A_1(u^e_h, v^e_h), w_1)_{\bar{\Omega}^e_{xy}} = &\int\limits_{\bar{\Omega}^e_{xy}} \left(-\,{}_s\sigma_{xx}\left(\frac{\partial w_1}{\partial x}\right) - {}_s\sigma_{yx}\left(\frac{\partial w_1}{\partial y}\right) + \frac{1}{2}\left(\frac{\partial(m_{xz})}{\partial x} + \frac{\partial(m_{yz})}{\partial y}\right)\frac{\partial w_1}{\partial y}\right) d\Omega \\ &+ \oint\limits_{\Gamma^e} \big({}_s\sigma_{xx}(n_x) + {}_s\sigma_{yx}(n_y)\big) w_1 \; d\Gamma \\ &- \frac{1}{2}\oint\limits_{\Gamma^e}\left(\frac{\partial(m_{xz})}{\partial x} + \frac{\partial(m_{yz})}{\partial y}\right) n_y w_1 \; d\Gamma \end{aligned} \tag{3.19}$$

$$\begin{aligned}(A_2(u^e_h, v^e_h), w_2)_{\bar{\Omega}^e_{xy}} = &\int\limits_{\bar{\Omega}^e_{xy}} \left(-\,{}_s\sigma_{xy}\left(\frac{\partial w_2}{\partial x}\right) - {}_s\sigma_{yy}\left(\frac{\partial w_2}{\partial y}\right) - \frac{1}{2}\left(\frac{\partial(m_{xz})}{\partial x} + \frac{\partial(m_{yz})}{\partial y}\right)\frac{\partial w_2}{\partial x}\right) d\Omega \\ &+ \oint\limits_{\Gamma^e} \big({}_s\sigma_{xy}(n_x) + {}_s\sigma_{yy}(n_y)\big) w_2 \; d\Gamma \\ &+ \frac{1}{2}\oint\limits_{\Gamma^e}\left(\frac{\partial(m_{xz})}{\partial x} + \frac{\partial(m_{yz})}{\partial y}\right) n_x w_2 \; d\Gamma \end{aligned} \tag{3.20}$$

Integrating by parts again for the moment terms in (3.19) and (3.20)

$$(A_1(u_h^e, v_h^e), w_1)_{\bar{\Omega}_{xy}^e} = \int\limits_{\bar{\Omega}_{xy}^e} \left(-\, {}_s\sigma_{xx} \left(\frac{\partial w_1}{\partial x}\right) - {}_s\sigma_{yx} \left(\frac{\partial w_1}{\partial y}\right) - \frac{1}{2} \left(m_{xz} \frac{\partial^2 (w_1)}{\partial x \partial y} + m_{yz} \frac{\partial^2 (w_1)}{\partial y^2} \right) \right) d\Omega$$
$$+ \oint\limits_{\Gamma^e} w_1 \left({}_s\sigma_{xx}(n_x) + {}_s\sigma_{yx}(n_y) - \frac{1}{2} \left(\frac{\partial (m_{xz})}{\partial x} + \frac{\partial (m_{yz})}{\partial y} \right) n_y \right) d\Gamma$$
$$+ \frac{1}{2} \oint\limits_{\Gamma_e} \frac{\partial w_1}{\partial y} \left(m_{xz}(n_x) + m_{yz}(n_y) \right) d\Gamma \tag{3.21}$$

$$(A_2(u_h^e, v_h^e), w_2)_{\bar{\Omega}_{xy}^e} = \int\limits_{\bar{\Omega}_{xy}^e} \left(-\, {}_s\sigma_{xy} \left(\frac{\partial w_2}{\partial x}\right) - {}_s\sigma_{yy} \left(\frac{\partial w_2}{\partial y}\right) + \frac{1}{2} \left(m_{xz} \frac{\partial^2 (w_2)}{\partial x^2} + m_{yz} \frac{\partial^2 (w_2)}{\partial y \partial x} \right) \right) d\Omega$$
$$+ \oint\limits_{\Gamma^e} w_2 \left({}_s\sigma_{xy}(n_x) + {}_s\sigma_{yy}(n_y) + \frac{1}{2} \left(\frac{\partial (m_{xz})}{\partial x} + \frac{\partial (m_{yz})}{\partial y} \right) n_x \right) d\Gamma$$
$$- \frac{1}{2} \oint\limits_{\Gamma_e} \frac{\partial w_2}{\partial x} \left(m_{xz}(n_x) + m_{yz}(n_y) \right) d\Gamma \tag{3.22}$$

Let

$$\begin{aligned} t_x &= {}_s\sigma_{xx}(n_x) + {}_s\sigma_{yx}(n_y) - \frac{1}{2}\left(\frac{\partial}{\partial x}(m_{xz}) + \frac{\partial}{\partial y}(m_{yz}) \right) n_y \\ t_y &= {}_s\sigma_{xy}(n_x) + {}_s\sigma_{yy}(n_y) + \frac{1}{2}\left(\frac{\partial}{\partial x}(m_{xz}) + \frac{\partial}{\partial y}(m_{yz}) \right) n_x \end{aligned} \tag{3.23}$$

and $\quad m_n = m_{xz}(n_x) + m_{yz}(n_y)$

Using (3.23) one can write (3.21) and (3.22) as follows

$$(A_1(u_h^e, v_h^e), w_1)_{\bar{\Omega}_{xy}^e} = \int\limits_{\bar{\Omega}_{xy}^e} \left(-\, {}_s\sigma_{xx} \left(\frac{\partial w_1}{\partial x}\right) - {}_s\sigma_{yx} \left(\frac{\partial w_1}{\partial y}\right) - \frac{1}{2} \left(m_{xz} \frac{\partial^2 (w_1)}{\partial x \partial y} + m_{yz} \frac{\partial^2 (w_1)}{\partial y^2} \right) \right) d\Omega$$
$$+ \oint\limits_{\Gamma^e} (w_1\ t_x) d\Gamma + \oint\limits_{\Gamma^e} \frac{\partial w_1}{\partial y} \left(\frac{m_n}{2} \right) d\Gamma \tag{3.24}$$

$$(A_2(u_h^e, v_h^e), w_2)_{\bar{\Omega}_{xy}^e} = \int\limits_{\bar{\Omega}_{xy}^e} \left(- {}_s\sigma_{xy} \left(\frac{\partial w_2}{\partial x} \right) - {}_s\sigma_{yy} \left(\frac{\partial w_2}{\partial y} \right) + \frac{1}{2} \left(m_{xz} \frac{\partial^2 (w_2)}{\partial x^2} + m_{yz} \frac{\partial^2 (w_2)}{\partial y \partial x} \right) \right) d\Omega$$
$$+ \oint\limits_{\Gamma^e} (w_2 \; t_y) d\Gamma + \oint\limits_{\Gamma^e} \frac{\partial w_2}{\partial x} \left(- \frac{m_n}{2} \right) d\Gamma \tag{3.25}$$

From (3.24) and (3.25) one can conclude that the primary and the secondary variables (PV and SV) are

PV	SV
u	t_x
v	t_y
$\frac{\partial u}{\partial y}$	$\frac{m_n}{2}$
$\frac{\partial v}{\partial x}$	$\frac{-m_n}{2}$

Substituting for ${}_s\sigma_{xx}$, ${}_s\sigma_{xy}$, ${}_s\sigma_{yy}$ from (3.4) into (3.24) and (3.25) and ${}_i\Theta_z$ from (3.9) into (3.5)-(3.8) and then (3.5)-(3.8) into (3.24) and (3.25) one can write

$$(A_1(u_h^e, v_h^e), w_1)_{\bar{\Omega}_{xy}^e} = B_1^e(u_h^e, v_h^e; w_1) - l_1^e(w_1) \tag{3.26}$$

$$(A_2(u_h^e, v_h^e), w_2)_{\bar{\Omega}_{xy}^e} = B_2^e(u_h^e, v_h^e; w_2) - l_2^e(w_2) \tag{3.27}$$

in which

$$B_1^e(u_h^e, v_h^e; w_1) = = \int\limits_{\bar{\Omega}_{xy}^e} \left\{ - \left(D_{11} \frac{\partial u_h^e}{\partial x} + D_{12} \frac{\partial v_h^e}{\partial y} \right) \frac{\partial w_1}{\partial x} - D_{33} \left(\frac{\partial u_h^e}{\partial y} + \frac{\partial v_h^e}{\partial x} \right) \frac{\partial w_1}{\partial y} \right.$$
$$\left. - \frac{1}{2} \left(\underset{\sim}{\alpha} - \underset{\sim}{\beta} \right) \left[\left(\frac{\partial^2 u_h^e}{\partial x \partial y} - \frac{\partial^2 v_h^e}{\partial x^2} \right) \frac{\partial^2 w_1}{\partial x \partial y} + \left(\frac{\partial^2 u_h^e}{\partial y^2} - \frac{\partial^2 v_h^e}{\partial y \partial x} \right) \frac{\partial^2 w_1}{\partial y^2} \right] \right\} d\Omega \tag{3.28}$$

$$l_1^e(w_1) = -\oint_{\Gamma^e} w_1 \; t_x \; d\Gamma - \oint_{\Gamma^e} \frac{\partial w_1}{\partial y} \left(\frac{m_n}{2}\right) d\Gamma \tag{3.29}$$

$$\begin{aligned} B_2^e(u_h^e, v_h^e; w_2) == \int_{\bar{\Omega}_{xy}^e} \Bigg\{ &- D_{33}\left(\frac{\partial u_h^e}{\partial y} + \frac{\partial v_h^e}{\partial x}\right)\frac{\partial w_2}{\partial x} - \left(D_{21}\frac{\partial u_h^e}{\partial x} + D_{22}\frac{\partial v_h^e}{\partial y}\right)\frac{\partial w_2}{\partial y} \\ &+ \frac{1}{2}\left(\underset{\sim}{\alpha} - \underset{\sim}{\beta}\right)\left[\left(\frac{\partial^2 u_h^e}{\partial x \partial y} - \frac{\partial^2 v_h^e}{\partial x^2}\right)\frac{\partial^2 w_2}{\partial x^2} + \left(\frac{\partial^2 u_h^e}{\partial y^2} - \frac{\partial^2 v_h^e}{\partial x \partial y}\right)\frac{\partial^2 w_2}{\partial y \partial x}\right]\Bigg\} d\Omega \end{aligned} \tag{3.30}$$

$$l_2^e(w_2) = -\oint_{\Gamma^e} w_2 \; t_y \; d\Gamma - \oint_{\Gamma^e} \frac{\partial w_2}{\partial x} \left(-\frac{m_n}{2}\right) d\Gamma \tag{3.31}$$

Functional $B_1^e(\cdot,\cdot)$, $B_2^e(\cdot,\cdot)$, $l_1^e(\cdot)$, $l_2^e(\cdot)$ are linear in all of their arguments. One notes that $l_1^e(\cdot)$, $l_2^e(\cdot)$ are concomitants resulting only as a consequence of integration by parts. This can be represented (3.28)-(3.31) in matrix and vector form

$$\begin{Bmatrix} (A_1(u_h^e, v_h^e), w_1)_{\bar{\Omega}_{xy}^e} \\ (A_2(u_h^e, v_h^e), w_2)_{\bar{\Omega}_{xy}^e} \end{Bmatrix} = \begin{Bmatrix} B_1^e(u_h^e, v_h^e; w_1) \\ B_2^e(u_h^e, v_h^e; w_2) \end{Bmatrix} - \begin{Bmatrix} l_1^e(w_1) \\ l_2^e(w_2) \end{Bmatrix} \tag{3.32}$$

$$= \begin{Bmatrix} b_{11}^e(u_h^e, w_1) + & b_{12}^e(v_h^e, w_1) \\ b_{21}^e(u_h^e, w_2) + & b_{22}^e(v_h^e, w_2) \end{Bmatrix} - \begin{Bmatrix} l_1^e(w_1) \\ l_2^e(w_2) \end{Bmatrix} \tag{3.33}$$

in which

$$\begin{aligned} b_{11}^e(u_h^e, w_1) = \int_{\bar{\Omega}_{xy}^e} \Bigg\{ &- \left(D_{11}\frac{\partial u_h^e}{\partial x}\;\frac{\partial w_1}{\partial x}\right) - D_{33}\left(\frac{\partial u_h^e}{\partial y}\;\frac{\partial w_1}{\partial y}\right) \\ &- \frac{1}{2}\left(\underset{\sim}{\alpha} - \underset{\sim}{\beta}\right)\left[\left(\frac{\partial^2 u_h^e}{\partial x \partial y}\;\;\frac{\partial^2 w_1}{\partial x \partial y}\right) + \left(\frac{\partial^2 u_h^e}{\partial y^2}\;\;\frac{\partial^2 w_1}{\partial y^2}\right)\right]\Bigg\} d\Omega \end{aligned} \tag{3.34}$$

$$
\begin{aligned}
b_{12}^e(v_h^e, w_1) = \int\limits_{\bar{\Omega}_{xy}^e} \Bigg\{ & - \left(D_{12} \frac{\partial v_h^e}{\partial y} \frac{\partial w_1}{\partial x} \right) - D_{33} \left(\frac{\partial v_h^e}{\partial x} \frac{\partial w_1}{\partial y} \right) \\
& - \frac{1}{2} \left(\underset{\sim}{\alpha} - \underset{\sim}{\beta} \right) \left[- \left(\frac{\partial^2 v_h^e}{\partial x^2} \; \frac{\partial^2 w_1}{\partial x \partial y} \right) - \left(\frac{\partial^2 v_h^e}{\partial y \partial x} \; \frac{\partial^2 w_1}{\partial y^2} \right) \right] \Bigg\} d\Omega
\end{aligned}
\tag{3.35}
$$

$$
\begin{aligned}
b_{21}^e(u_h^e, w_2) = \int\limits_{\bar{\Omega}_{xy}^e} \Bigg\{ & - \left(D_{33} \frac{\partial u_h^e}{\partial y} \frac{\partial w_2}{\partial x} \right) - D_{21} \left(\frac{\partial u_h^e}{\partial x} \frac{\partial w_2}{\partial y} \right) \\
& + \frac{1}{2} \left(\underset{\sim}{\alpha} - \underset{\sim}{\beta} \right) \left[\left(\frac{\partial^2 u_h^e}{\partial x \partial y} \; \frac{\partial^2 w_2}{\partial x^2} \right) + \left(\frac{\partial^2 u_h^e}{\partial y^2} \; \frac{\partial^2 w_2}{\partial y \partial x} \right) \right] \Bigg\} d\Omega
\end{aligned}
\tag{3.36}
$$

$$
\begin{aligned}
b_{22}^e(v_h^e, w_2) = \int\limits_{\bar{\Omega}_{xy}^e} \Bigg\{ & - \left(D_{33} \frac{\partial v_h^e}{\partial x} \frac{\partial w_2}{\partial x} \right) - D_{22} \left(\frac{\partial v_h^e}{\partial y} \frac{\partial w_2}{\partial y} \right) \\
& + \frac{1}{2} \left(\underset{\sim}{\alpha} - \underset{\sim}{\beta} \right) \left[- \left(\frac{\partial^2 v_h^e}{\partial x^2} \; \frac{\partial^2 w_2}{\partial x^2} \right) - \left(\frac{\partial^2 v_h^e}{\partial x \partial y} \; \frac{\partial^2 w_2}{\partial y \partial x} \right) \right] \Bigg\} d\Omega
\end{aligned}
\tag{3.37}
$$

The functions b_{ij}^e; $i, j = 1, 2$ have the following properties.

$$
\left.
\begin{array}{c}
b_{11}^e(u_h^e, w_1) = b_{11}^e(w_1, u_h^e) \Rightarrow b_{11}^e(\cdot, \cdot) \;\; \text{is symmetric} \\
b_{22}^e(v_h^e, w_2) = b_{22}^e(w_2, v_h^e) \Rightarrow b_{22}^e(\cdot, \cdot) \;\; \text{is symmetric} \\
b_{12}^e(w_2, u_h^e) = b_{21}^e(u_h^e, w_2) \\
b_{21}^e(w_1, v_h^e) = b_{12}^e(v_h^e, w_1)
\end{array}
\right\}
\tag{3.38}
$$

Equation (3.32) are the weak form of the mathematical model (3.12)and (3.13). Equations (3.38) imply that the element equations constructed from (3.38) using local approximation u_h^e and v_h^e will contain symmetric element coefficient matrix.

Let $\bar{\Omega}_{xy}^e$ be a nine node p-version hierarchical element with local approximation in higher order scalar product space $H^{k,p}(\bar{\Omega}_{xy}^e)$ [84–87]. Consider $\bar{\Omega}_{xy}^e \rightarrow \bar{\Omega}_{\xi\eta}^e = [-1, 1]$ x $[-1, 1]$, a map of $\bar{\Omega}_{xy}^e$ in ξ, η space in a two unit square. Then, one can write

$$
u_h^e(\xi, \eta) = \sum_{i=1}^{n^u} N_i^u(\xi, \eta)({}^u\delta_i^e)
\tag{3.39}
$$

$$v_h^e(\xi,\eta) = \sum_{i=1}^{n^v} N_i^v(\xi,\eta)(^v\delta_i^e) \tag{3.40}$$

$N_i^u(\xi,\eta)$ and $N_i^v(\xi,\eta)$ are local approximation functions and $(^u\delta_i^e)$ and $(^v\delta_i^e)$ are nodal degrees of freedom for u and v. Using (3.39) and (3.40)

$$\begin{aligned} w_1 &= \delta u_h^e = N_j^u(\xi,\eta); \quad j = 1,2,...,n^u \\ w_2 &= \delta v_h^e = N_k^v(\xi,\eta); \quad k = 1,2,...,n^v \end{aligned} \tag{3.41}$$

Let the total degrees of freedom for an element e be $\{\delta^e\}$

$$\{\delta^e\} = \{^u\delta^e\} \cup \{^v\delta^e\} = \begin{Bmatrix} \{^u\delta^e\} \\ \{^v\delta^e\} \end{Bmatrix} \tag{3.42}$$

Substituting from (3.39)-(3.41) into (3.34)-(3.37), one can write

$$\begin{aligned} b_{11}^e = \int\limits_{\bar{\Omega}_{xy}^e} \Bigg\{ &- D_{11}\left(\sum_{i=1}^{n^u} \frac{\partial N_i^u}{\partial x}(^u\delta_i^e)\right)\frac{\partial N_j^u}{\partial x} - D_{33}\left(\sum_{i=1}^{n^u} \frac{\partial N_i^u}{\partial y}(^u\delta_i^e)\right)\frac{\partial N_j^u}{\partial y} \\ &- \frac{1}{2}(\underset{\sim}{\alpha} - \underset{\sim}{\beta})\left[\left(\sum_{i=1}^{n^u} \frac{\partial^2 N_i^u}{\partial x \partial y}(^u\delta_i^e)\right)\frac{\partial^2 N_j^u}{\partial x \partial y} + \left(\sum_{i=1}^{n^u} \frac{\partial^2 N_i^u}{\partial y^2}(^u\delta_i^e)\right)\frac{\partial^2 N_j^u}{\partial y^2}\right]\Bigg\} d\Omega\,; \quad j = 1,2,\dots,n^u \end{aligned} \tag{3.43}$$

$$\begin{aligned} b_{12}^e = \int\limits_{\bar{\Omega}_{xy}^e} \Bigg\{ &- D_{12}\left(\sum_{i=1}^{n^v} \frac{\partial N_i^v}{\partial y}(^v\delta_i^e)\right)\frac{\partial N_j^u}{\partial x} - D_{33}\left(\sum_{i=1}^{n^v} \frac{\partial N_i^v}{\partial y}(^v\delta_i^e)\right)\frac{\partial N_j^u}{\partial y} \\ &+ \frac{1}{2}(\underset{\sim}{\alpha} - \underset{\sim}{\beta})\left[\left(\sum_{i=1}^{n^v} \frac{\partial^2 N_i^v}{\partial x^2}(^v\delta_i^e)\right)\frac{\partial^2 N_j^u}{\partial x \partial y} + \left(\sum_{i=1}^{n^v} \frac{\partial^2 N_i^v}{\partial y \partial x}(^v\delta_i^e)\right)\frac{\partial^2 N_j^u}{\partial y^2}\right]\Bigg\} d\Omega\,; \quad j = 1,2,\dots,n^u \end{aligned} \tag{3.44}$$

$$b^e_{21} = \int\limits_{\bar{\Omega}^e_{xy}} \Bigg\{ -D_{33}\left(\sum_{i=1}^{n^u}\frac{\partial N^u_i}{\partial y}({}^u\delta^e_i)\right)\frac{\partial N^v_j}{\partial x} - D_{21}\left(\sum_{i=1}^{n^u}\frac{\partial N^u_i}{\partial x}({}^u\delta^e_i)\right)\frac{\partial N^v_j}{\partial y}$$
$$+\frac{1}{2}(\underset{\sim}{\alpha}-\underset{\sim}{\beta})\Bigg[\left(\sum_{i=1}^{n^u}\frac{\partial^2 N^u_i}{\partial x\partial y}({}^u\delta^e_i)\right)\frac{\partial^2 N^v_j}{\partial x^2} + \left(\sum_{i=1}^{n^u}\frac{\partial^2 N^u_i}{\partial y^2}({}^u\delta^e_i)\right)\frac{\partial^2 N^v_j}{\partial y\partial x}\Bigg]\Bigg\}d\Omega\;; \quad j=1,2,\dots,n^v \tag{3.45}$$

$$b^e_{22} = \int\limits_{\bar{\Omega}^e_{xy}} \Bigg\{ -D_{33}\left(\sum_{i=1}^{n^v}\frac{\partial N^v_i}{\partial x}({}^v\delta^e_i)\right)\frac{\partial N^v_j}{\partial x} - D_{22}\left(\sum_{i=1}^{n^v}\frac{\partial N^v_i}{\partial y}({}^v\delta^e_i)\right)\frac{\partial N^v_j}{\partial y}$$
$$-\frac{1}{2}(\underset{\sim}{\alpha}-\underset{\sim}{\beta})\Bigg[\left(\sum_{i=1}^{n^v}\frac{\partial^2 N^v_i}{\partial x^2}({}^v\delta^e_i)\right)\frac{\partial^2 N^v_j}{\partial x^2} + \left(\sum_{i=1}^{n^v}\frac{\partial^2 N^v_i}{\partial x\partial y}({}^v\delta^e_i)\right)\frac{\partial^2 N^v_j}{\partial x\partial y}\Bigg]\Bigg\}d\Omega\;; \quad j=1,2,\dots,n^v \tag{3.46}$$

Using (3.43)-(3.46) one can write (3.33) as follows

$$\begin{Bmatrix} (A_1(u^e_h, v^e_h), w_1)_{\bar{\Omega}^e_{xy}} \\ (A_2(u^e_h, v^e_h), w_2)_{\bar{\Omega}^e_{xy}} \end{Bmatrix} = -[K^e]\{\delta^e\} + \{P^e\}$$
$$= -\begin{bmatrix} [{}^{11}K^e] & [{}^{12}K^e] \\ [{}^{21}K^e] & [{}^{22}K^e] \end{bmatrix}\begin{Bmatrix} \{{}^u\delta^e\} \\ \hdashline \{{}^v\delta^e\} \end{Bmatrix} + \begin{Bmatrix} \{P^e_1\} \\ \hdashline \{P^e_2\} \end{Bmatrix} \tag{3.47}$$

in which

$${}^{11}K^e_{ij} = \int\limits_{\bar{\Omega}^e_{xy}} \Bigg\{D_{11}\frac{\partial N^u_i}{\partial x}\;\frac{\partial N^u_j}{\partial x} + D_{33}\frac{\partial N^u_i}{\partial y}\;\frac{\partial N^u_j}{\partial y} + \frac{1}{2}(\underset{\sim}{\alpha}-\underset{\sim}{\beta})\Bigg[\frac{\partial^2 N^u_i}{\partial x\partial y}\;\frac{\partial^2 N^u_j}{\partial x\partial y}$$
$$+\frac{\partial^2 N^u_i}{\partial y^2}\;\frac{\partial^2 N^u_j}{\partial y^2}\Bigg]\Bigg\}d\Omega\;; \quad i,j=1,2,\dots,n^u \tag{3.48}$$

$${}^{12}K^e_{ij} = \int\limits_{\bar{\Omega}^e_{xy}} \Bigg\{D_{12}\frac{\partial N^u_i}{\partial x}\;\frac{\partial N^v_j}{\partial y} + D_{33}\frac{\partial N^u_i}{\partial y}\;\frac{\partial N^v_j}{\partial x} - \frac{1}{2}(\underset{\sim}{\alpha}-\underset{\sim}{\beta})\Bigg[\frac{\partial^2 N^u_i}{\partial x\partial y}\;\frac{\partial^2 N^v_j}{\partial x^2}$$
$$+\frac{\partial^2 N^u_i}{\partial y^2}\;\frac{\partial^2 N^v_j}{\partial y\partial x}\Bigg]\Bigg\}d\Omega\;; \quad i=1,2,...,n^u\;;\; j=1,2,\dots,n^v \tag{3.49}$$

$$
\begin{aligned}
{}^{21}K_{ij}^{e} = \int\limits_{\bar{\Omega}_{xy}^{e}} \Bigg\{ & D_{33}\frac{\partial N_i^v}{\partial x}\,\frac{\partial N_j^u}{\partial y} + D_{21}\frac{\partial N_i^v}{\partial y}\,\frac{\partial N_j^u}{\partial x} - \frac{1}{2}(\underset{\sim}{\alpha} - \underset{\sim}{\beta})\Bigg[\frac{\partial^2 N_i^v}{\partial x^2}\,\frac{\partial^2 N_j^u}{\partial x \partial y} \\
& + \frac{\partial^2 N_i^v}{\partial y \partial x}\,\frac{\partial^2 N_j^u}{\partial y^2}\Bigg]\Bigg\} d\Omega \; ; \qquad i = 1,2,\dots,n^u \; ; \; j = 1,2,\dots,n^v
\end{aligned}
\tag{3.50}
$$

$$
\begin{aligned}
{}^{22}K_{ij}^{e} = \int\limits_{\bar{\Omega}_{xy}^{e}} \Bigg\{ & D_{33}\frac{\partial N_i^v}{\partial x}\,\frac{\partial N_j^v}{\partial x} + D_{22}\frac{\partial N_i^v}{\partial y}\,\frac{\partial N_j^v}{\partial y} + \frac{1}{2}(\underset{\sim}{\alpha} - \underset{\sim}{\beta})\Bigg[\frac{\partial^2 N_i^v}{\partial x^2}\,\frac{\partial^2 N_j^v}{\partial x^2} \\
& + \frac{\partial^2 N_i^v}{\partial x \partial y}\,\frac{\partial^2 N_j^v}{\partial x \partial y}\Bigg]\Bigg\} d\Omega \; ; \qquad i,j = 1,2,\dots,n^v
\end{aligned}
\tag{3.51}
$$

One notes that

$$
\begin{gathered}
{}^{11}K_{ij}^{e} = {}^{11}K_{ji}^{e} \; ; \qquad i,j = 1,2,\dots,n^u, \text{hence } \left[{}^{11}K^e\right] \text{ is symmetric} \\
{}^{12}K_{ij}^{e} = {}^{21}K_{ji}^{e} \; ; \; \left[{}^{12}K^e\right] = \left[{}^{21}K^e\right]^T \\
{}^{22}K_{ij}^{e} = {}^{22}K_{ji}^{e} \; ; \qquad i,j = 1,2,\dots,n^u, \text{hence } \left[{}^{22}K^e\right] \text{ is symmetric}
\end{gathered}
\tag{3.52}
$$

From (3.52 one can conclude that $[K^e]$ in (3.47) is symmetric. For the entire discretization one can write

$$
\begin{Bmatrix} (A_1(u_h^e, v_h^e), w_1)_{\bar{\Omega}_{xy}^e} \\ (A_2(u_h^e, v_h^e), w_2)_{\bar{\Omega}_{xy}^e} \end{Bmatrix} = \sum_e \begin{Bmatrix} (A_1(u_h^e, v_h^e), w_1)_{\bar{\Omega}_{xy}^e} \\ (A_2(u_h^e, v_h^e), w_2)_{\bar{\Omega}_{xy}^e} \end{Bmatrix} = -\sum_e \big([K^e]\{\delta^e\} + \{P^e\} \big) = 0
\tag{3.53}
$$

Hence,

$$
\sum_e \big([K^e]\{\delta^e\} \big) = \sum_e \{P^e\}
\tag{3.54}
$$

or

$$
[K]\{\delta\} = \{P\}
\tag{3.55}
$$

in which

$$
[K] = \sum [K^e]; \quad \text{assembly of element equations}
\tag{3.56}
$$

$$
\{\delta\} = \cup_e \{\delta^e\}
\tag{3.57}
$$

$$\{P\} = \sum_{e} \{P^e\} \tag{3.58}$$

3.3 Approximation spaces and some remarks

1. Since the mathematical model ((3.12) and (3.13) contains up to fourth order derivatives of the displacements, the approximation functions in spaces $V_h \subset H^{k,p}(\bar{\Omega}^e_{xy})$, $k \geq 5$ are admissible in (3.12) and (3.13) and $k = 5$ i.e. local approximation of class $C^4(\bar{\Omega}^e_{xy})$ corresponds to minimally conforming space.

2. Weak form (3.32) resulting from GM/WF only contains derivatives of up to order two of u and v, hence it is tempting to use u^e_h and v^e_h of class $C^1(\bar{\Omega}^e_{xy})$ but in doing so one can rely on weak convergence of the solutions of class C^1 to class C^2 and eventually to class C^4 needed for the mathematical model.

3. Numerical values of the coefficients of $[K^e]$ are obtained using Gauss quadrature.

4. Assembled equations (3.55) for $\bar{\Omega}^T_{xy}$ are solved after imposing boundary conditions.

5. Linearity of the algebraic system and symmetry $[K^e]$ and $[K]$ are due to the fact that the differential operator in (3.12) and (3.13) is linear in displacements u and v and the adjoint $\boldsymbol{A}^*$ of the differential operator $\boldsymbol{A}$ is same (when the mathematical model is expressed in displacements u and v)

6. In the study of the model problem one choses $\underset{\sim}{\beta} = 0$ (based on the material presented in [72]) i.e. one considers balance of moments of moments as a balance law, hence the Cauchy moment tensor is symmetric.

3.4 A least squares formulation in $\mathbb{R}^2$ (plane stress) based on residual functional

One considers the following mathematical model (obtained using (3.1)-(3.10)) in the dimensionless form (in the absence of balance of moments of moments as a balance law [72]) consisting of first order partial differential equations.

$$
\begin{gathered}
\frac{\partial_s\sigma_{xx}}{\partial x}+\frac{\partial_s\sigma_{yx}}{\partial y}+\frac{\partial_a\sigma_{yx}}{\partial y}=0; \quad \frac{\partial_s\sigma_{xy}}{\partial x}+\frac{\partial_s\sigma_{yy}}{\partial y}-\frac{\partial_a\sigma_{yx}}{\partial x}=0 \\
\frac{\partial m_{xz}}{\partial x}+\frac{\partial m_{yz}}{\partial y}+2({}_a\sigma_{yx})=0
\end{gathered} \tag{3.59}
$$

$$
{}_s\sigma_{xx}=D_{11}\frac{\partial u}{\partial x}+D_{12}\frac{\partial u}{\partial y}; \quad {}_s\sigma_{yy}=D_{21}\frac{\partial u}{\partial x}+D_{22}\frac{\partial v}{\partial y}; \quad {}_s\sigma_{xy}=D_{33}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) \tag{3.60}
$$

$$
\begin{aligned}
{}_sm_{xz}&=\left(\frac{E_0}{m_0L_0}\right)\underset{\sim}{\alpha}\frac{\partial({}_i\Theta_z)}{\partial x}; \quad {}_sm_{yz}=\left(\frac{E_0}{m_0L_0}\right)\underset{\sim}{\alpha}\frac{\partial({}_i\Theta_z)}{\partial y} \\
{}_am_{xz}&=\left(\frac{E_0}{m_0L_0}\right)\underset{\sim}{\beta}\frac{\partial({}_i\Theta_z)}{\partial x}; \quad {}_am_{yz}=\left(\frac{E_0}{m_0L_0}\right)\underset{\sim}{\beta}\frac{\partial({}_i\Theta_z)}{\partial y}
\end{aligned} \tag{3.61}
$$

$$
\begin{gathered}
D_{11}=D_{22}=\frac{E}{1-\nu^2}; \quad D_{12}=D_{21}=\frac{\nu E}{1-\nu^2}; \quad D_{33}=G=\frac{E}{2(1+\nu)} \\
{}_i\Theta_z=\frac{1}{2}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)
\end{gathered} \tag{3.62}
$$

In (2.128)-(2.131), $\frac{E_0}{m_0L_0}$ is in fact one, but it has been left in constitutive theory for the moment tensors for sake of clarity. Equations (3.59)-(3.62) are a system of eleven first order linear coupled differential equations in eleven dependent variables u, v, ${}_s\sigma_{xx}$, ${}_s\sigma_{yy}$, ${}_s\sigma_{xy}$, ${}_a\sigma_{yx}$, ${}_sm_{xz}$, ${}_sm_{yz}$, ${}_am_{xz}$, ${}_am_{yz}$ and ${}_i\Theta_z$. A least square formulation (LSF) of (2.128)-(2.131) is constructed using residual functionals [84, 88–93] resulting from each of the eleven equations when their local approximations are substituted in them. The local approximations considered in higher order scalar product space $H^{k,p}(\bar{\Omega}^e)$, $\bar{\Omega}^e$ being an element of the discretization which are p-version hierarchical with higher order global differentiability. Since (3.59)-(3.62) are a system of first order equations $k=2$ i.e. local approximations of class $C^1(\bar{\Omega}^e)$ for each variable constitute minimally conforming space or approximations [84]. However, for the model problems considered here the solutions are sufficiently smooth, thus permitting the use of $C^0(\bar{\Omega}^e)$ local approximations with weak convergence to $C^1(\bar{\Omega}^e)$.

3.5 Model Problems

In this section one can consider three model problems in $\mathbb{R}^2$: 1) Simply supported thin plate with transverse in plane loading. 2) fixed-fixed thin plate with transverse in plane loading. 3) a square plate with a circular hole at the centre subjected to uniaxial uniform loading.

Remarks.

(1) In all numerical studies one considers both formulations, GM/WF as well as LSP.

(2) One chooses $\beta = 0$ in all studies [72] which implies that ${}_a\boldsymbol{m} = 0$ and $\boldsymbol{m} = {}_s\boldsymbol{m}$ implying that balance of moments of moments is a balance law. This is necessary for incorporating correct polar physics due to internal rotations in the mathematical model [72]. Thus, ${}_am_{xz}$ and ${}_am_{yz} = 0$ in (3.59)-(3.62) and the mathematical model reduces to nine partial differential equations in nine dependent variables.

(3) One notes that the integral form in GM/WF contains upto second order derivatives of u and v, hence $k = 3$ is minimally conforming approximation space (i.e. solutions of class C^2 in x and y) for the integral forms for which all integrals over the spatial discretization are Riemann in the other hand for $k = 2$ i.e. C^1 approximations in x and y, the integrals over the spatial discretization are Lebesgue. For simply supported and fixed-fixed plate one considers numerical studies with $k = 3, p = 5$ (i.e. C^2 local approximations in x and y with p level of 5) and with $k = 2$, $p = 7$ (i.e. C^1 local approximations with p-level of 7). In case of square plate with a hole one considers $k = 2$ with p-level of 7.

(4) Computations for least square formulations are only performed for the simple supported and

fixed-fixed plate to provide comparisons with the solutions obtained using GM/WF. In these studies one chooses $k = 1, p = 9$ i.e. solutions of class C^0 with p-level of nine as used in references [5, 72]. For $k = 1$ integrals over the discretization is in Lebesgue sense.

3.5.1 Simple supported and fixed-fixed plate: Model problems 1 and 2

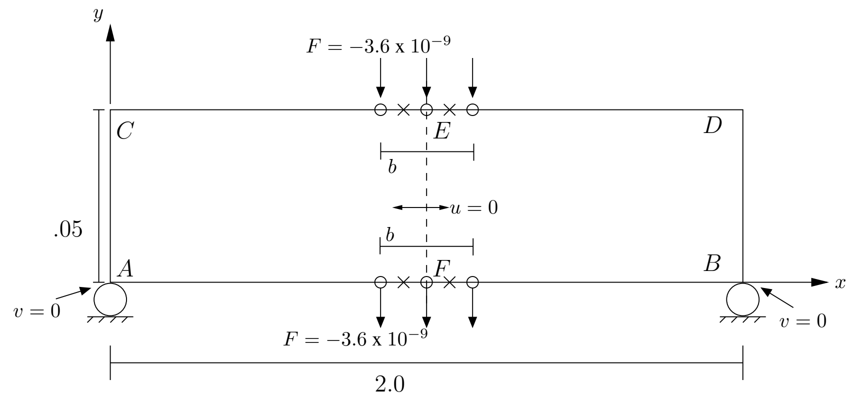
One considers a thin plate of length $\hat{l} = 20$ inches with width $\hat{b} = 0.5$ inches and thickness $\hat{t} = 0.1$ inches. With $L_0 = 10$ inches, the dimensionless plate is 2 x 0.05 x 0.01. Figure 3.1 (a) and (b) show schematics of the plate, boundary conditions and loading for the formulation based on GM/WF for both simple supported and fixed-fixed plate. The load is applied over a length of $w = 0.4$ as forces at the nodes that corresponds to uniform stress in the y-direction. Figures 3.2 (a) and (b) show same schematics with BC's and loading used in least squares formulation. In all numerical studies the plates are discretized using a 20 element uniform discretization (10 elements along the length and two elements width b) using a nine node p-version hierarchical higher order global differentiability finite elements. In all computations one chooses $\hat{E} = E_0 = 30$ x 10^6 psi, hence $E = 1$ and one varies dimensionless $\underset{\sim}{\alpha}$ between 0 - 0.25 with $\underset{\sim}{\beta} = 0.0$. $\underset{\sim}{\alpha} = 0$ corresponds to classical continuum theory. Progressively increasing values of $\underset{\sim}{\alpha}$ produces progressively more pronounced polar physics (resistance to deformation). Numerical solutions ae calculated for the following:

(1) For GM/WF one considers $k = 3$ (solutions of class C^2 in x and y) with $p = 5$. For this integrals over the spatial discretization are Riemann

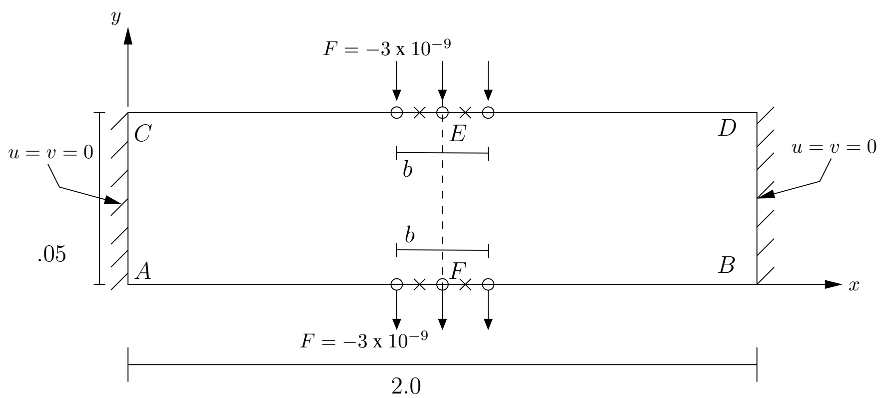
(2) For GM/WF one also considers $k = 2$ (solutions of class C^1 in x and y) with $p = 7$. For this choice integrals over the spatial discretization are in Lebesgue sense.

(3) Comparing the computed solutions from (1) and (2) one confirms that both solutions are almost indistinguishable from each other.

(4) For Least squares formulation one considers solutions of class C^0 in x and y with p-level of nine [5, 72]



(a) Simply suppported plate ($b = 0.4$)



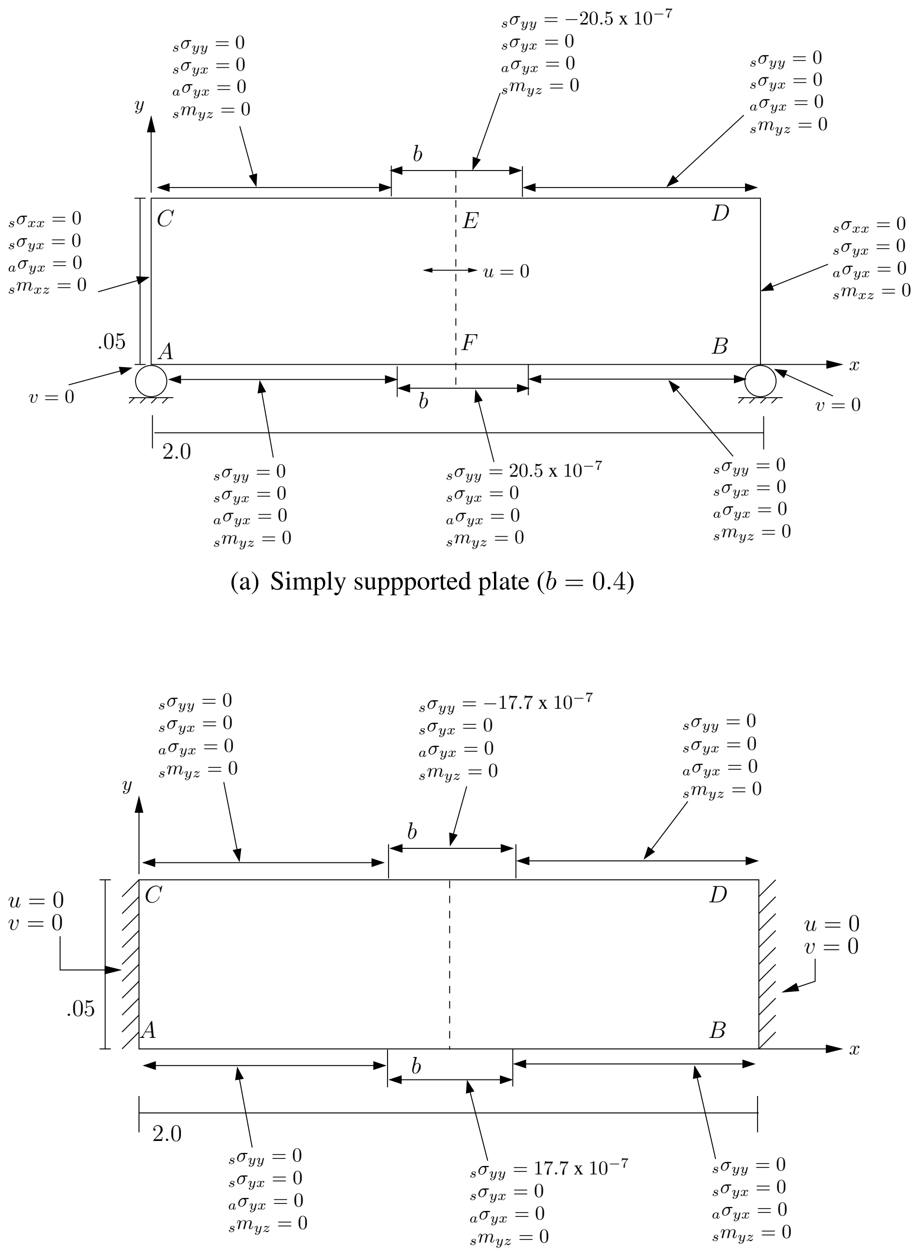
(b) Fixed-fixed plate ($b = 0.4$)

Figure 3.1: Model Problem 1 and 2: Schematics, BCs and loading (dimensionless) : GM/WF

Results.

GM/WF:

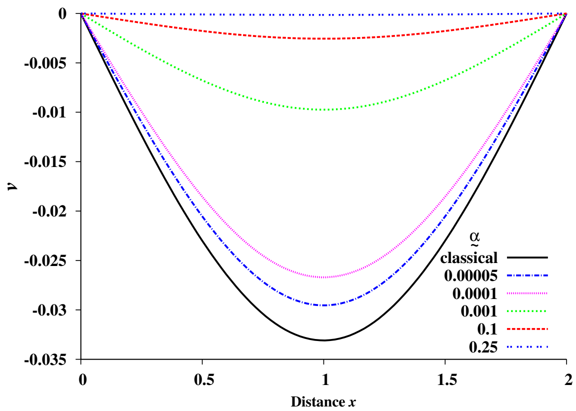
Figures 3.3 (a)-(c) shows plots of $v, {}_i\Theta_z$ and ${}_sm_{xz}$ versus x and y = 0.025 (center line of the plate) for $\underset{\sim}{\alpha} = 0 - 0.25$. For $\underset{\sim}{\alpha} = 0$ one gets classical continuum behavior. Progressively increasing values of $\underset{\sim}{\alpha}$ results in progressively increasing resistance to deformation, hence reducing displacement v, reducing rotation ${}_i\Theta_z$ but increasing moment ${}_sm_{xz}$. Similar graphs of v, ${}_i\Theta_z$ and ${}_sm_{xz}$ versus x at $y = 0.025$ for fixed-fixed plate are shown in figures 3.4 (a)-(c) for $\underset{\sim}{\alpha} = 0 - 0.25$. One observes similar trends in the behaviors of v, ${}_i\Theta_z$ and ${}_sm_{xz}$ with increasing $\underset{\sim}{\alpha}$ values. Due to fixed boundaries the displacement v is significantly reduced and ${}_i\Theta_z$ and ${}_sm_{xz}$ follow accordingly.



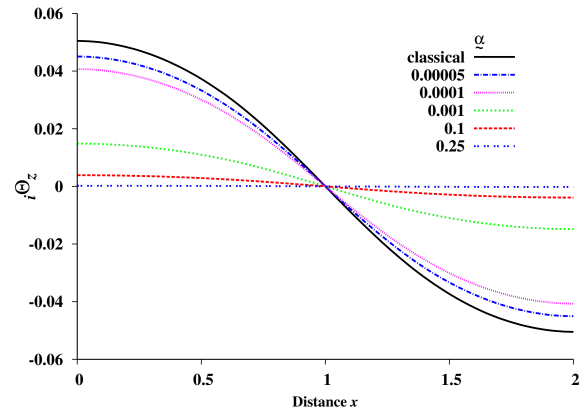
(a) Simply suppported plate ($b = 0.4$)

(b) Fixed-fixed plate ($b = 0.4$)

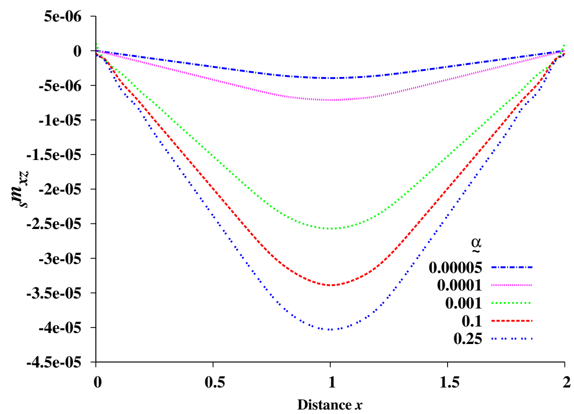
Figure 3.2: Model Problem 1 and 2: Schematics, BCs and loading (dimensionless) : LSP



(a) v versus x at $y = 0.025$



(b) $_i\Theta_z$ versus x at $y = 0.025$

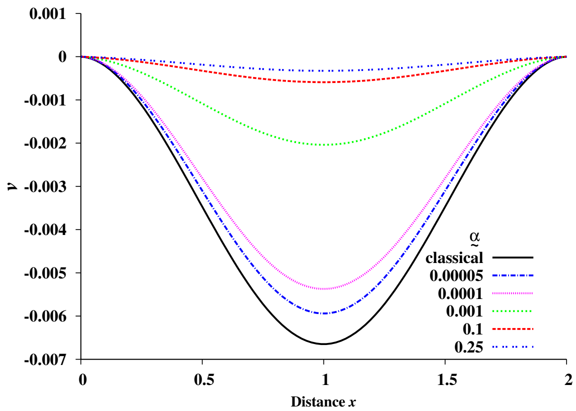


(c) $_sm_{xz}$ versus x at $y = 0.025$

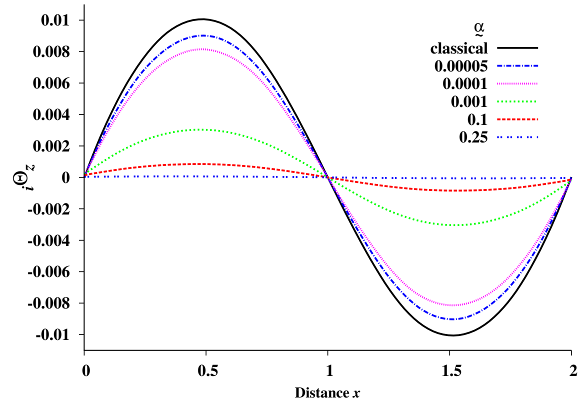
Figure 3.3: Graphs of $v, {}_i\Theta_z$ and $_sm_{xz}$ versus $y = 0.025$: Simply supported plate (GM/WF)

Comparison of results: GM/WF and LSP:

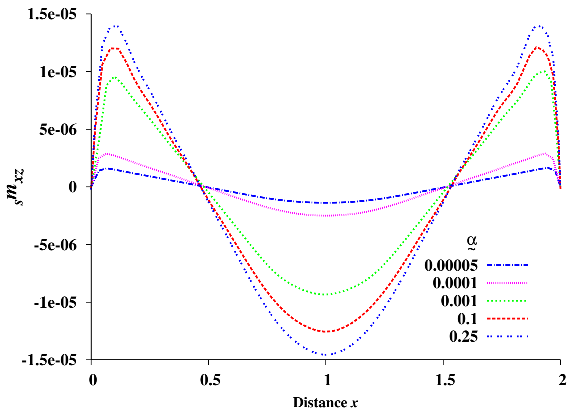
The numerical solutions obtained from LS formulation for exactly same BCs and loading (figure 3.2) using local approximations of class C^0 at p-level 9 are compared with those obtained using GM/WF (C^2 solutions at $p = 5$ or C^1 solutions at $p = 7$). In the LSP the residual functional for the discretization is of the order $O(10^{-15})$. This ensures that the computed solutions satisfy the governing differential equations in the pointwise sense, hence the computed solutions are virtually same as the theoretical solution. Comparison of these solutions with GM/WF provides a check on the accuracy of the solutions obtained using GM/WF.



(a) v versus x at $y = 0.025$



(b) $_i\Theta_z$ versus x at $y = 0.025$



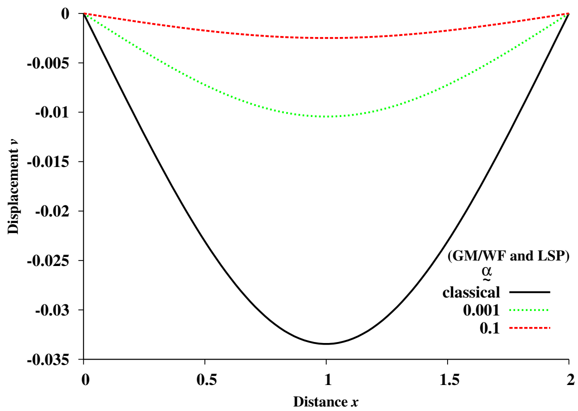
(c) $_sm_{xz}$ versus x at $y = 0.025$

Figure 3.4: Graphs of $v, {}_i\Theta_z$ and $_sm_{xz}$ versus $y = 0.025$: Fixed-fixed plate (GM/WF)

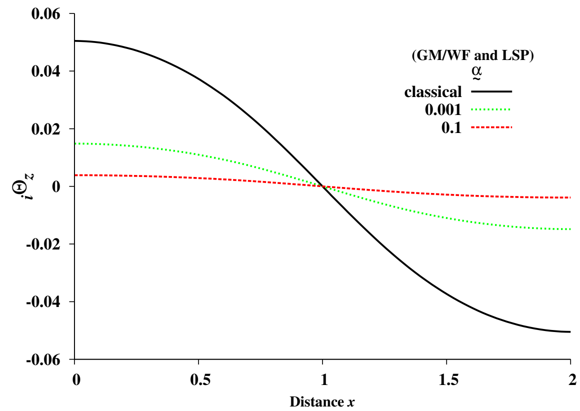
Figures 3.5 (a)-(c) show the plots of v, $_i\Theta_z$ and $_sm_{xz}$ versus x at $y = 0.025$ for $\underset{\sim}{\alpha} = 0, 0.001$ and 0.1 obtained using GM/WF and a comparison with least squares method for simple supported plate. Similar results for GM/WF and a comparison with LSP for fixed-fixed plate are shown in figures 3.6 (a)-(c). In both figures (3.5) and (3.6), v, $_i\Theta_z$ and $_sm_{xz}$ obtained using GM/WF and LSP are in perfect agreement with each other for all three values of $\underset{\sim}{\alpha}$.

3.5.2 A square plate with a circular hole: Model problem 3

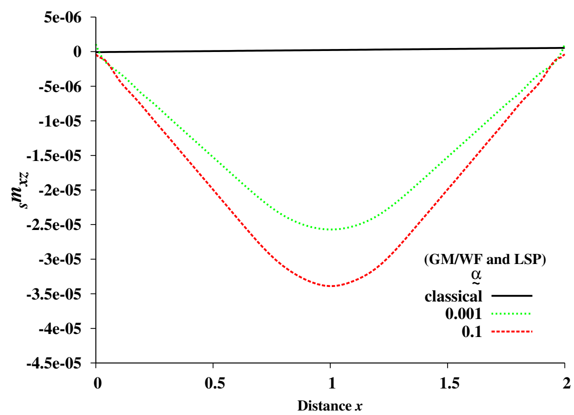
One considers a 6"x 6" square plate of thickness 0.1" with a 0.48" diameter circular hole at the center. $L_0 = 1.5$" is used. The material properties, reference quantities etc. used here are same as for model problems 1 and 2. Poisson's ratio of 0.3 is used. This gives rise to a 4 x 4 x 0.06



(a) v versus x at $y = 0.025$



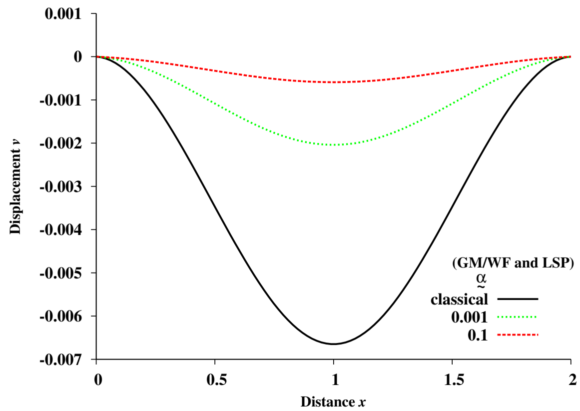
(b) ${}_{i}\Theta_z$ versus x at $y = 0.025$



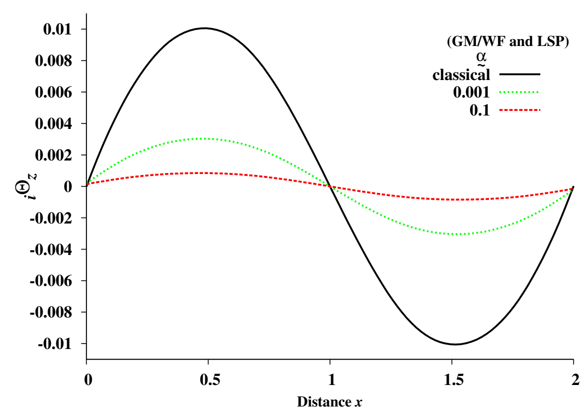
(c) ${}_{s}m_{xz}$ versus x at $y = 0.025$

Figure 3.5: Simply supported plate: comparison of LSP and GM/WF

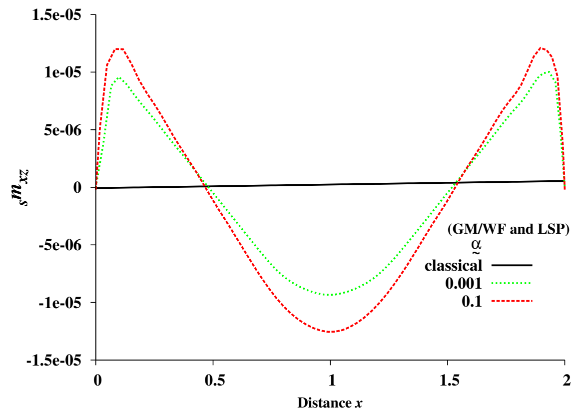
dimensionless plate with a hole diameter of 0.16 (figure 3.7(a)). The plate is subjected to uniform displacement of 0.01 (dimensionless) on its vertical faces that creates a uniform dimensionless stress field of $(\sigma_{xx})_0 = 0.0048$. The details of the BC's and loading for quarter plate are shown in figure 3.7(a). Figure 3.7(c) shows a graded discretization of the quarter plate. The plate is divided in four bicubic patches (figure 3.7(b)). In each patch a 3 x 3 uniform discretization of nine-node p-version hierarchical elements with higher order global differentiability local approximation [86,94] is used giving a total of 36 elements for the quarter of the plate. Computations are performed only using the formulation based on GM/WF with local approximation of class C^1 i.e $k = 2$ in x and y space (for distorted elements in $\mathbb{R}^2$, xy-space [84, 94]) with p-level of 7. For this choice of k,



(a) v versus x at $y = 0.025$



(b) ${}_i\Theta_z$ versus x at $y = 0.025$



(c) ${}_s m_{xz}$ versus x at $y = 0.025$

Figure 3.6: Fixed-fixed plate: comparison of LSP and GM/WF

the integrals over the discretization are Lebesgue, but due to smoothness of the solution of one can expect these solutions to converge to class C^2 in the weak sense.

Results:

The stresses ${}_s\sigma_{xx}$ and ${}_s\sigma_{yy}$ are normalized using $(\sigma_{xx})_0$, $({}_s\sigma_{xx})_n = {}_s\sigma_{xx}/(\sigma_{xx})_0$ and $({}_s\sigma_{yy})_n = {}_s\sigma_{yy}/(\sigma_{xx})_0$. It is well known that based on classical continuum theory (when $\alpha = 0$) stress concentration is 3.0 at E (figure 3.7(a)) i.e. in this case one can expect $({}_s\sigma_{xx})_n = 3.0$ at point E (figure 3.7(a)). With increasing values of α increasing presence of internal polar physics is present, hence we observe progressively increasing resistance to deformation. As a consequence stresses

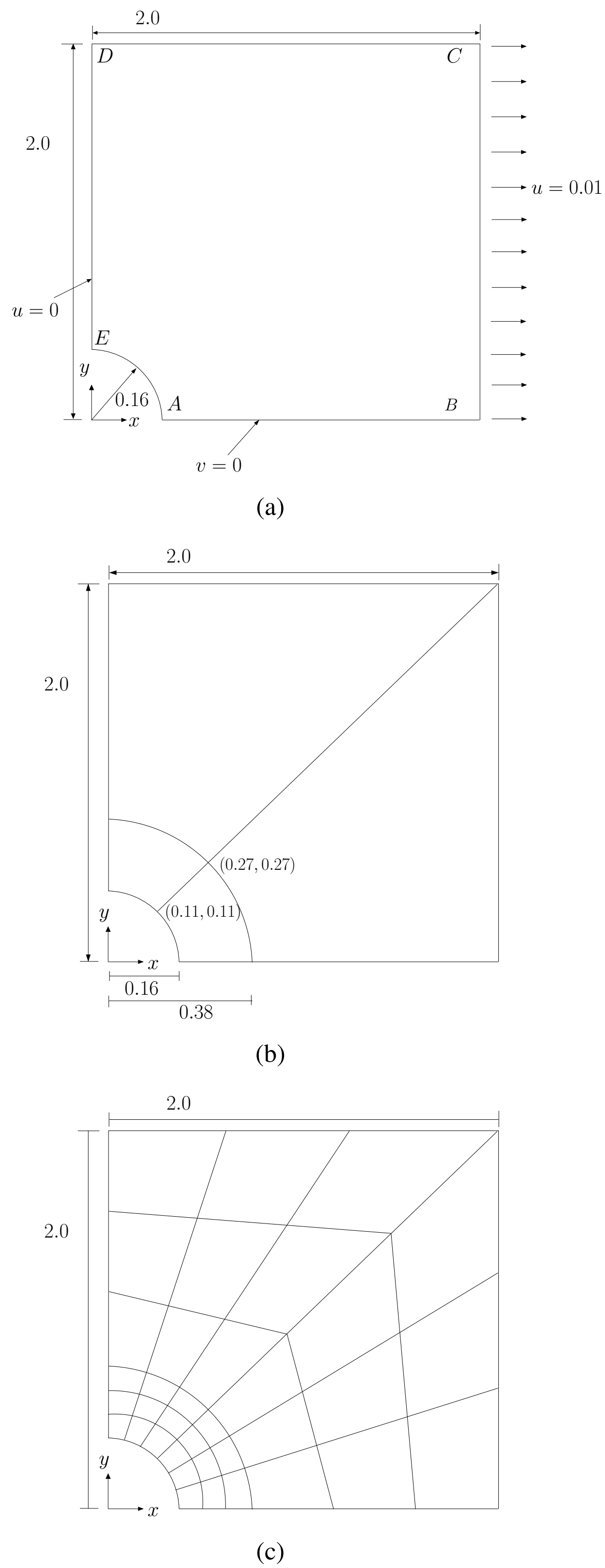
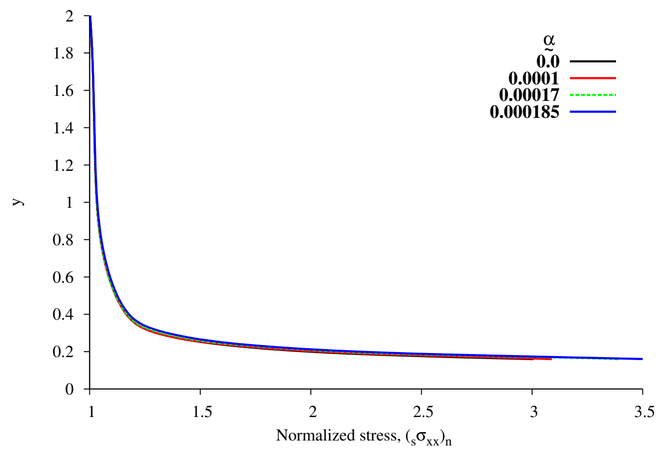
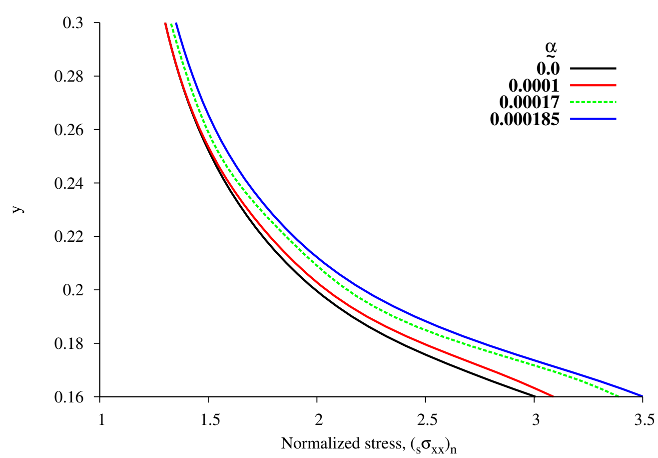


Figure 3.7: Schematic of a quarter of the plate with a circular hole, BCs, loading and finite element discretization

increase as α increases. Figure 3.8(a) shows plots of $({}_s\sigma_{xx})_n$ versus y along the the edge of ED of the plate for different values of α. At D, $({}_s\sigma_{xx})_n = 1$ as expected. As one approaches E from D, stress $({}_s\sigma_{xx})_n$ increases. The exploded view in the vicinity of point E shown in figure 3.8(b) confirms that when $\alpha = 0$ i.e. the classical theory, $({}_s\sigma_{xx})_n$ is indeed 3.0. With increasing α (for range considered here) $({}_s\sigma_{xx})_n$ as high as 3.5 is obtained. From figure 3.8(a) one notes that the for progressively increasing values of α, $({}_s\sigma_{xx})_n$ progressively increases from a value of 1.0 at D to upto 3.5 at E for the largest value of α ($\alpha = 0.000185$) used in the studies.



(a) Normalized stress versus y at $x = 0.0$



(b) Exploded view in the vicinity of hole

Figure 3.8: Normalized stress versus y at $x = 0.0$

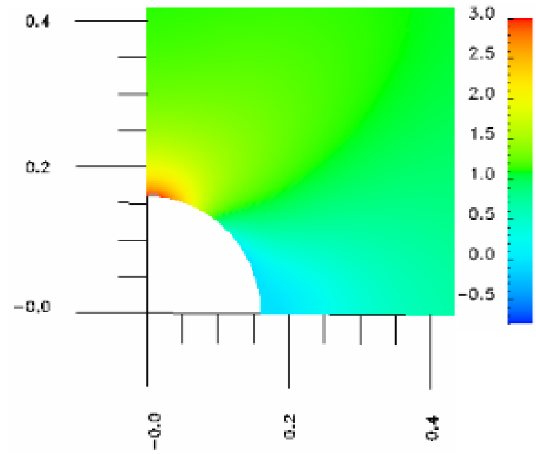
Figures 3.9(a)-(c) show carpet plots of $({}_s\sigma_{xx})_n$ for $\alpha = 0.0, 0.0001, 0.000185$. With progressively increasing values of α, higher values of $({}_s\sigma_{xx})_n$ in the entire quarter of the plate are observed compared to classical theory ($\alpha = 0.0$), most significant increase being at point E as expected.

3.6 General Remark

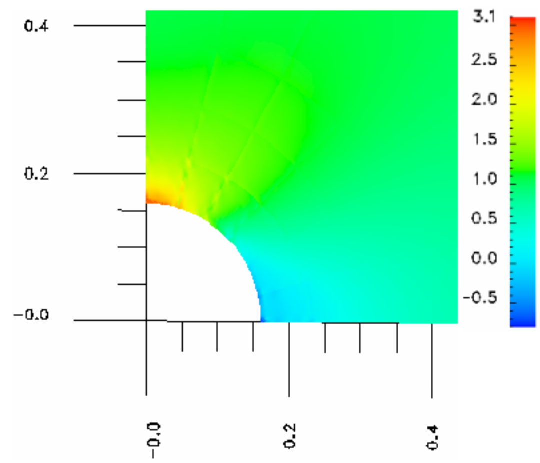
In this section one can make some remarks related to the two finite element formulations (GM/WF, LSP) in context with numerical studies.

(1) It is obvious that for the model problems (in $\mathbb{R}^2$) the GM/WF has only two dependent variables u and v whereas LSP based on first order system of PDEs has nine dependent variables resulting in enormous computational inefficiency but permitting flexibility to use C^0 local approximations.

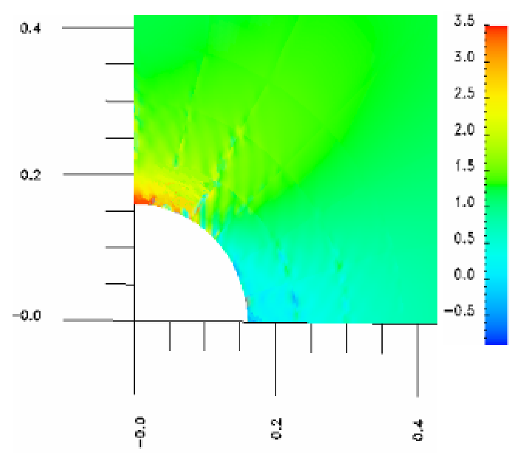
(2) In LSP there is no concept of secondary variables as in GM/WF, hence there are no self equilibrating quantities in LS finite element formulation. As a result, all dependent variable pertaining to the known physics must be specified on the boundaries of the domain. For example in GM/WF stress free boundaries are automatically satisfied due to sum of secondary variables being zero. Same is true for moment free boundaries. However, in LSP all boundary information must be defined in the problem data (figures 3.1(a),(b) for GM/WF and figures 3.2(a),(b) for LSP containing schematics and BCs for model problem 1 and 2 clearly illustrates this). Due to the need of defining all dependent variable specifications on the boundaries of the domain in LSP as BCs, often the specifications become cumbersome and non-conclusive due to redundancies in their specification. GM/WF is completely free of such problems. In case of square plate with a circular hole, the situation is much more difficult in LSP as in this case the stress and moment components normal to the hole are zero while the tangential components need to be computed. Definition of such BCs require either constrained equations or a rotated local coordinate system on the hole boundary that is normal and tangent to the hole boundary. In GM/WF normal tractions (both stress and



(a) classical



(b) $\alpha = 0.0001$



(c) $\alpha = 0.000185$

Figure 3.9: Contour plots showing quarter of a plate with a circular hole under uniaxial loading

moment) are secondary variables, hence their sum on the hole boundary in naturally zero at each node, thus this condition is automatically satisfied.

(3) In GM/WF as well as in LSP one considers integrals over the spatial discretization in Lebesgue sense, but there is no issue of their convergence. LS residual functional $O(10^{-15})$ and perfect match of GM/WF results with LSP for model problems 1 and 2 confirm that both GM/WF and LSP results are sufficiently converged to be as good as theoretical solutions.

3.7 Conclusions

In this chapter the mathematical model consisting of conservation an balance law in Lagrangian description for non-classical continuum theory for elastic solids (small strain small deformation physics without dissipation and memory) incorporating internal rotation physics due to displacement gradient tensor are considered (derived in reference [72]). In such solids the deformation physics due to mechanical work is reversible, hence the differential operator $\boldsymbol{A}$ in these mathematical models when expressed purely in terms of displacements are such that their adjoint $\boldsymbol{A}^*$ is same as $\boldsymbol{A}$. Thus, in such mathematical models GM/WF is ideal for the finite element formulation of the corresponding BVP. One makes the following specific remarks and observations and draws some conclusions from the work presented in this chapter.

(1) GM/WF is ideal for reversible processes as in the present case. In such mathematical models $\boldsymbol{A}^* = \boldsymbol{A}$ holds.

(2) LSP with first order system of PDEs are computationally non competitive with GM/WF. In the work presented here GM/WF has only two dependent variables whereas LSP has nine.

(3) An important question is “could one have used LSP" for the mathematical model in displacements u and v derived for GM/WF. Of course one could but: (1) this would require solutions of class C^4 or of class C^3 for sure (2) stress and moment boundary conditions(zero or non zero) are extremely difficult to define as the stresses and moments are not dependent

variables any more in the mathematical model. (3) Due to lack of secondary variable, zero stress and moment boundary conditions also need to be specified. Due to these difficulties it is perhaps more convenient to use mathematical models consisting of first order PDEs in LSP.

(4) Numerical studies for the three model problems clearly demonstrates superiority of GM/WF over LSP in almost all aspects.

(5) Numerical solutions computed using GM/WF and those using LSP satisfy PDEs, almost in the pointwise sense as the residual functional for the discretization is $O(10^{-15})$.

(6) Presence of increasing polar physics with increasing α is clearly demonstrated in model problem 1 and 2 (also shown in references [5, 72] using finite element formulation based on LSP) using both finite element formulations.

(7) The third model problem is rather difficult to study using finite element formulation based on LSP due to the difficulty of specifying zero boundary conditions of stress and moment normal to the hole boundary. In GM/WF the secondary variables and their sum being zero on free boundaries automatically satisfies these BCs.

(8) In the plate problem with a circular hole the stress concentration at point E of 3.0 is predicted correctly when $\alpha = 0$. With progressively increasing α, the stress concentration at E increases from 3 to 3.5 for the largest value of $\alpha = 0.000185$ used here.

(9) The finite element formulation based on GM/WF for non-classical continuum models is a valuable approach for the mathematical models in which the deformation due to mechanical work is reversible. The finite element formulation based on GM/WF for non-classical continuum models presented here is superior and meritorious in all aspects when compared to the finite element formulations based on least square processes. The only disadvantage one could possibly point out is the presence of up to fourth order derivatives of displacements in

the mathematical models. In view of the research work on k-version [[85–87]] this is hardly of any consequence.

Chapter 4

Recommendations for future work

The balance of moments of moments balance law shown to be essential in case of non-classical continuum theories for elastic solids (without dissipation and memory) also needs to be considered and evaluated in case of non-classical thermoviscoelastic solids with and without memory for its necessity and its consistency with regard to other balance laws specially for second law of thermodynamics.

While the construction of the integral form based on GM/WF in the finite element processes for thermoelastic non-classical solids with reversible deformation due to mechanical work has been shown to be meritorious and advantageous compared to the integral forms based on LSP using residual functional, investigation of the possibility of the use of GM/WF in the mathematical models for thermoelastic non-classical solids with and without memory is natural extension of the work presented here.

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